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# A Method for Finding Roots of Arbitrary Matrices

**1. Jacobi's Method for Real Symmetric Matrices.** There is a well known method due to JACOBI[1] for diagonalizing real symmetric matrices. It consists of performing a sequence of orthogonal transformations (rotations), each one on a two-dimensional subspace of the underlying vector space in which the matrix is considered to be defined. Thus, if we call the original matrix $A$, we transform with an orthogonal matrix $S_{(km)}$ as follows:

$$B = S^T_{(km)} A S_{(km)}. \tag{1}$$

(The superscript $T$ indicates the transposed matrix.) The main idea of Jacobi's method is to annihilate one of the off-diagonal elements ($A_{km}$) by this rotation. After we have annihilated $A_{km}$ with this rotation, we go to element $A_{k'm'}$ and find a transformation $S_{(k'm')}$ which will annihilate $A_{k'm'}$. This last transformation, if it affects $A_{km}$, will in general "deannihilate" it. Nevertheless, one selects each off-diagonal element in turn and performs a transformation which will annihilate it. After one has covered every off-diagonal element once, one has performed what we will call an *iteration*. The essence of Jacobi's method is that even though one undoes the annihilation (in general) of all the off-diagonal elements except the last one, with a sufficient number of iterations the off-diagonal elements will converge to zero, leaving a diagonal matrix. One obtains the final transformation matrix by post-multiplying by the $S_{(km)}$ at each annihilation, starting with the unit matrix. Thus:

$$S_{\text{final}} = S^{(1)}_{k_1m_1} S^{(1)}_{k_1m_2} \cdots S^{(1)}_{k_Rm_R} S^{(2)}_{k_1m_1} \cdots S^{(N)}_{k_Rm_R} \tag{2}$$

where the superscript refers to the number of the iteration and $R$ is equal to the number of off-diagonal elements ($= \frac{1}{2}n(n-1)$ for an $n$th order matrix). This $S$ has the property that:

$$S^T A S = D \tag{3}$$

where $D$ is the diagonal matrix whose elements are the roots of the matrix. With each root, there corresponds an eigenvector, which is the column of the $S$-matrix which occupies the same position in the $S$-matrix as the column containing the eigenvalue in question occupies in the $D$ matrix.

**2. Generalization for Arbitrary Matrices.** There are various fundamental theorems which show that a real symmetric (or, for that matter, Hermitian) matrix has real roots and can always be diagonalized by an orthogonal transformation of some kind.

These theorems all break down for non-Hermitian matrices, real or otherwise. In fact, it is known that there exist matrices which no *collineatory* transformation can diagonalize. These are all degenerate, and the best reduction that can be effected is to the Jordan canonical form. However, if we give up the ideal of

***diagonalizing*** matrices, and restrict ourselves to ***triangularizing*** them, then we have at our disposal a theorem of SCHUR[2] which states that an ***arbitrary*** matrix can be triangularized by ***unitary*** transformations. Therefore, instead of specifying $S_{(km)}$ to be orthogonal, we shall require it to be merely unitary, and perform the following transformation on the arbitrary complex matrix, $A$:

$$B = S^*_{(km)} A S_{(km)}. \tag{4}$$

The elements of $S_{(km)}$ are now complex, and it is in general impossible to characterize $S$ by one parameter. There are really ***two*** parameters necessary to characterize the transformation. We shall define the $2 \times 2$ non-trivial submatrix of $S_{(km)}$ as follows:

$$\begin{bmatrix} s_{kk} & s_{km} \\ s_{mk} & s_{mm} \end{bmatrix} = \begin{bmatrix} a & -\bar{c} \\ c & a \end{bmatrix} \tag{5}$$

where $a$ is real and positive, and the following relation holds:

$$a^2 + |c|^2 = 1. \tag{6}$$

It can be shown that there is no loss in generality in thus restricting $S_{(km)}$.

The elements of the transformed matrix are given by:

$$\begin{aligned} B_{ik} &= aA_{ik} + cA_{im} \\ B_{im} &= -\bar{c}A_{ik} + aA_{im} \end{aligned} \tag{7}$$

$$\begin{aligned} B_{kj} &= aA_{kj} + \bar{c}A_{mj} \\ B_{mj} &= -cA_{kj} + aA_{mj} \end{aligned} \tag{8}$$

$$\begin{aligned} B_{kk} &= a^2A_{kk} + |c|^2A_{mm} + acA_{km} + a\bar{c}A_{mk} \\ B_{km} &= a^2A_{km} - \bar{c}^2A_{mk} + a\bar{c}(A_{mm} - A_{kk}) \\ B_{mk} &= a^2A_{mk} - c^2A_{km} + ac(A_{mm} - A_{kk}) \\ B_{mm} &= a^2A_{mm} + |c|^2A_{kk} - acA_{km} - a\bar{c}A_{mk}. \end{aligned} \tag{9}$$

In view of the non-hermiticity of $A$ it is in general not possible to make both $B_{mk}$ and $B_{km}$ vanish. However, it is always possible to make $B_{mk}$ alone vanish, since there are three equations in the three unknowns $a$, $Rl(c)$, $Im(c)$ if we bear in mind that setting $B_{mk} = 0$ gives two equations and that (6) must be satisfied.

By setting:

$$a = [1 + |\mu|^2]^{-\frac{1}{2}} \tag{10a}$$

$$c = \mu[1 + |\mu|^2]^{-\frac{1}{2}} \equiv \mu a \tag{10b}$$

we can reduce the number of unknowns and equations to two. We then have, for $\mu$:

$$A_{km}\mu^2 - (A_{mm} - A_{kk})\mu - A_{mk} = 0; \tag{11}$$

hence (setting $A_{mm} - A_{kk} \equiv 2\Delta_{mk}$), we obtain:

$$\mu = \Delta_{mk} \pm \sqrt{\Delta^2_{mk} + A_{km}A_{mk}}. \tag{12}$$

Of these two roots, we choose the one of smaller modulus, in order to avoid large rotations when possible. (This can be done without much trouble in a digital program.)

There are properties of these unitary transformations which are of importance, viz.:

$$|B_{ik}|^2 + |B_{im}|^2 = |A_{ik}|^2 + |A_{im}|^2 \tag{13}$$

$$|B_{kj}|^2 + |B_{mj}|^2 = |A_{kj}|^2 + |A_{mj}|^2 \tag{14}$$

$$|B_{kk}|^2 + |B_{km}|^2 + |B_{mk}|^2 + |B_{mm}|^2 = |A_{kk}|^2 + |A_{km}|^2 + |A_{mk}|^2 + |A_{mm}|^2. \tag{15}$$

It is no longer possible, as in the real symmetric case, to prove that the sum of squares of absolute values (S.S.A.V.) of the off-diagonal elements decreases. What is necessary to prove is that the S.S.A.V. of the sub-diagonal elements decreases to zero. It is not even true that this sum decreases monotonically.[8]

An example to illustrate this is the following:

$$A \equiv \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{16}$$

One may annihilate the lower left element with the transformation:

$$S_{(31)} \equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{17}$$

but the result is:

$$S^T_{(31)} A S_{(31)} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{18}$$

and we see that the S.S.A.V. of the sub-diagonal elements has increased from 1 to 2. However, one simply keeps on transforming away sub-diagonal elements without regard to the fluctuations of the S.S.A.V. of these elements. Finally they should decrease to zero and the process converge.

The justification for the last statement is based on "experimental" evidence, gained by triangularizing a variety of matrices with a program written in the "Speedcoding" floating-point system for the IBM 701. Floating-point is unnecessary in this problem and slows the machine by a factor of more than ten, but this program was written originally as much to test Speedcoding as to test triangularization. The average time per multiplication or addition in this system is 4 milliseconds.

With a series of matrices whose elements (both real and imaginary parts) were picked at random, the following are the number of iterations and times required for convergence to 10 decimal places (actually, with floating point, to $10^{-30}$ for sub-diagonal elements).

These data, though incomplete, indicate that the number of iterations in all probability does not increase more strongly than the first power of the order of

TABLE I. *Time for Convergence*

| Order | Time per Iteration | Number of Iterations |
|---|---|---|
| 2 | 2 | 3 |
| 3 | 5 | 6 |
| 4 | 10 | 9 |
| 5 | 20 | 12 |
| 6 | 40 | 14 |
| 7 | 55 | 14 |
| 8 | 85 | 17 |

the matrix. For degenerate matrices, the number of iterations increases considerably, rising to about 25 for a fourth order matrix with four equal roots. It is noteworthy, also, that the roots (which are, of course, the diagonal elements when triangularization is complete) are not accurate. This is because the unitary transformation really preserves the invariants of the matrix, i.e., the coefficients in its characteristic equation, and these do not specify the roots uniquely because of rounding error. For example, let $\lambda_0$ be a double root of:

$$(\lambda - \lambda_0)^2 = \lambda^2 - 2\lambda_0\lambda + \lambda_0^2 = 0. \tag{19}$$

What we are given originally, in general, is the expanded form of this equation. Now let us substitute for $(\lambda_0, \lambda_0)$ the pair $(\lambda_0 + \epsilon, \lambda_0 - \epsilon)$. The characteristic equation is then:

$$(\lambda - \lambda_0 - \epsilon)(\lambda - \lambda_0 + \epsilon) = \lambda^2 - 2\lambda_0\lambda + \lambda_0^2 - \epsilon^2 = 0. \tag{20}$$

Now if $|\lambda_0| \sim 1$ and $|\epsilon| \sim 10^{-5}$, then the characteristic equation in (20) will differ from that in (19) by $\epsilon^2$, or a quantity $\sim 10^{-10}$ which is of the same order of magnitude as the rounding error in a ten-digit machine. Double precision operations will not increase the accuracy, since the coefficients of the characteristic equation are not known with more than single accuracy. The roots, as we see, may be specified with an ambiguity much greater than the least count of the machine. In general, if there are $m$ equal roots and $D$ digits, the roots will be accurate to about $D/m$ digits. Therefore, it is in general impossible to decide numerically whether roots are equal or just very closely spaced.

These remarks do not apply to Hermitian matrices, where equal roots will come out equal within the actual rounding error of the machine. There is evidently some connection between the "degree of Hermiticity" of a matrix and the accuracy with which its repeated roots can be found.

It is of interest at this point to indicate the complications that arise in attempting to find the eigenvectors of the triangularized matrix, which distinguish markedly the Hermitian and non-Hermitian cases. In particular, it will be shown that, when a root is repeated, it is sometimes impossible to obtain the full number of eigenvectors corresponding to that root. First we look at the Hermitian case.

The original eigenvalue equation was:

$$A\psi^{(k)} = \alpha^{(k)}\psi^{(k)} \tag{21}$$

which now becomes:

$$D\varphi^{(k)} = \alpha^{(k)}\varphi^{(k)} \tag{22}$$

with

$$\psi^{(k)} = S\varphi^{(k)}. \tag{23}$$

Since $D_{ij} = \alpha^{(i)}\delta_{ij}$ we have:

$$\sum_j \alpha^{(i)}\delta_{ij}\varphi_j^{(k)} = \alpha^{(i)}\varphi_i^{(k)} = \alpha^{(k)}\varphi_i^{(k)}. \tag{24}$$

Hence, $\varphi_i^{(k)}$ must be equal to $\delta_{ki}$, i.e., all components of $\varphi^{(k)}$ vanish except the $k$th. Hence also, from (23):

$$\psi_i^{(k)} = \sum_j S_{ij}\varphi_j^{(k)} = \sum_j S_{ij}\delta_{kj} = S_{ik}. \tag{25}$$

This shows why $\psi^{(k)}$ is the $k$th column of $S$.

Now, in the case where $A$ is arbitrary, we can at most triangularize it to obtain:

$$S^*AS = T \tag{26}$$

where $T$ is a triangular matrix. Analogously to the Hermitian case, the transformed eigenvalue equation is:

$$T\varphi^{(k)} = \alpha^{(k)}\varphi^{(k)} \tag{27}$$

which is written out as:

$$\sum_{j=i}^{N} T_{ij}\varphi_j^{(k)} = \alpha^{(k)}\varphi_i^{(k)}. \tag{28}$$

We subtract one side from the other to obtain:

$$\sum_{j=i}^{N} (T_{ij} - \alpha^{(k)}\delta_{ij})\varphi_j^{(k)} = 0. \tag{29}$$

In eq. (24), we had in place of $T_{ij}$, $\alpha^{(i)}\delta_{ij}$. This gave $(\alpha^{(i)} - \alpha^{(k)})\varphi_i^{(k)} = 0$. Here, however, we have terms other than the diagonal ones coming in, so the situation is not so straightforward. Eq. (29) is, in fact, a triangular system of equations for $\varphi_j^{(k)}$ which must be solved for each value of $k$. Let us take $k = N$, so that the last diagonal element vanishes. Let also $i = N$ and consider the last equation, which is:

$$(\alpha^{(N)} - \alpha^{(N)})\varphi_N^{(N)} = 0. \tag{30}$$

Since the first factor vanishes, $\varphi_N^{(N)}$ is arbitrary. Hence we may solve the triangular system for all the other components of $\varphi^{(N)}$ in terms of $\varphi_N^{(N)}$ (assuming no degeneracy, i.e., that there is no other vanishing diagonal term). Next we set $k = N - 1$. The last equation of the system (29) is:

$$(\alpha^{(N)} - \alpha^{(N-1)})\varphi_N^{(N-1)} = 0. \tag{31}$$

Since the first factor does not vanish (by the assumption of no degeneracy), the second factor must. The $(N - 1)$th equation of (29) is:

$$(\alpha^{(N-1)} - \alpha^{(N-1)})\varphi_{N-1}^{(N-1)} + T_{N-1,N}\varphi_N^{(N-1)} = 0. \tag{32}$$

Here, the second term vanishes, and so does the first factor of the first term. Hence $\varphi_{N-1}^{(N-1)}$ is arbitrary. We may then solve for the other components of $\varphi^{(N-1)}$ in terms of $\varphi_{N-1}^{(N-1)}$. Similarly, we may calculate all the vectors $\varphi^{(k)}$ in terms of arbitrary scale factors, which may be chosen so as to normalize the (transformed) eigenvectors $\varphi^{(k)}$. Finally, from $\psi^{(k)} = S\varphi^{(k)}$, we obtain all the eigenvectors of the original problem.

When we do have degeneracy, let us, for definiteness, place one of the repeated roots in the last position of the diagonal. Now we proceed as above, except that for some value $M$ (say) of the index $k$ we will come across a diagonal coefficient in our triangular system of linear equations which vanishes just as the last diagonal element did. Two possibilities are now open. Either the rest of the linear combination (exclusive of the diagonal term) which constitutes the $M$th equation vanishes, or it does not. If the former is true, then we are entitled to choose for $\varphi_M^{(N)}$ another arbitrary number and hence obtain another eigenvector. If the latter is true, then we are forced to make it equal to zero by setting $\varphi_N^{(N)} = 0$. This causes us to lose one of the basis vectors of the degenerate subspace defined by the repeated root. We may then start as before by setting $\varphi_M^{(N)}$ equal to some arbitrary number and solve for the rest of the components of $\varphi^N$ as before. The case when we lose one of the eigenvectors belonging to an eigenvalue corresponds to the case in the Jordan canonical form when a 1 appears attached to two equal roots. When there is no 1 attached to a repeated root, we then have the full complement of eigenvectors for that root. Similar considerations apply to roots of higher multiplicity than 2.

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[1] See, for example, R. T. GREGORY, *MTAC*, v. 7, 1953, p. 215.
[2] F. MURNAGHAN, *Theory of Group Representations*. Johns Hopkins Univ. Press. Chap. 1.
[3] Unless a suggestion of J. von NEUMANN (private communication) is followed, according to which one attempts to minimize the S.S.A.V. of the sub-diagonal elements in the $m$th row and $k$th column between the pivotal subdiagonal element and the diagonal (including the pivotal element itself). This method, however, is much more time-consuming than that proposed in this paper.

# Double Interpolation Formulae and Partial Derivatives in Terms of Finite Differences

**1. Abstract.** For making interpolations at different parts of a table, double interpolation formulae for mixed forward, backward, and central differences have been derived. However, these formulae are rather cumbersome and time consuming in use. For interpolation with more than one variable, formulae in terms of the tabular entries $f_{ij}$ directly instead of differences are much simpler to use. SALZER[1] has derived such a formula for double-forward interpolation in terms of $f_{ij}$. This formula works satisfactorily for interpolation near the head of a table. For the purpose of interpolating at other parts of a table, double interpolation formulae for other than double-forward interpolation formula have been derived.

Expressions for partial derivatives of different orders in terms of both double

differences and $f_{ij}$ directly, including those needed in the numerical solution of partial differential equations, are also worked out.

**2. Double Interpolation Formulae.** For making a double interpolation, the tabular values are usually given in a square or rectangular region. Let us divide this region into nine smaller regions as indicated in Fig. 1 by Roman numerals. In order to interpolate values in all nine regions, nine different interpolation

| | | |
|---|---|---|
| II | VII | IV |
| VIII | V | IX |
| I | VI | III |

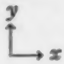


FIG. 1. A tabular values block

formulae are needed. These nine formulae can be further classified into five types, i.e., double-forward, semi-forward, double-backward, semi-central, and double-central interpolation formulae, as shown in Table 1. The double interpolation formulae now available are mostly given as functions of double differences. The interpolation formula in terms of double forward differences is known as Biermann's formula or the double Gregory-Newton formula. There are three double

TABLE I. *Different types of Interpolation Formulae*

| Region | Interpolation Formula | Type |
|---|---|---|
| I | Forward $x$, Forward $y$ | Double Forward |
| II | Forward $x$, Backward $y$ | Semi-Forward |
| III | Backward $x$, Forward $y$ | |
| IV | Backward $x$, Backward $y$ | Double Backward |
| V | Central Differences | Double Central |
| VI | Central $x$, Forward $y$ | Semi-Central |
| VII | Central $x$, Backward $y$ | |
| VIII | Forward $x$, Central $y$ | |
| IX | Backward $x$, Central $y$ | |

central difference formulae: Stirling-Stirling, Bessel-Bessel and Stirling-Bessel. The other types of the double interpolation formulae as functions of mixed double differences may also be derived easily.

However, these formulae in terms of double differences are rather cumbersome and time consuming in use. For interpolation with more than one variable, formulae in terms of the tabular entries $f_{ij}$ directly instead of double differences should be much simpler to use. Salzer[1,2] has derived such a formula for double forward interpolation in terms of $f_{ij}$ directly as given in the following equation

$$f(u, v) = \sum_{i+j=0}^{n} \binom{n-u-v}{n-i-j}\binom{u}{i}\binom{v}{j} f_{ij} \tag{1}$$

where $u = \dfrac{x - x_0}{h}$, $v = \dfrac{y - y_0}{k}$ and $h$ and $k$ are the tabular intervals.

$$\binom{u}{i} \text{ denotes } \frac{u(u-1)\cdots(u-i+1)}{i!} \text{ with } \binom{u}{0} = 1.$$

This formula works satisfactorily for interpolation near the head of a table or in region 1. For the purpose of interpolation in other regions, similar double interpolation formulae of $f_{ij}$ for other than double forward interpolation formula are required.

It is found that by correctly changing the signs of $u$ and $v$ as well as the corresponding subscripts of $f_{ij}$ in Salzer's formula, the interpolation formulae which apply in regions II, III, and IV can be obtained. For example, in order to get a semi-forward interpolation formula which applies in region II, i.e., forward $x$ and backward $y$, one simply substitutes $v$ into $-v$ and $f_{ij}$ into $f_{i,-j}$ in equation (1), and obtains

$$f(u, -v) = \sum_{i+j=0}^{n} \binom{n-u+v}{n-i-j}\binom{u}{i}\binom{-v}{j} f_{i,-j}.$$

Similarly, the other semi-forward interpolation formula which applies in region III can be obtained by substituting $-u$ by $u$, and $f_{ij}$ by $f_{-i,j}$ in equation (1). If one changes both $u$ to $-u$ and $v$ to $-v$ and at the same time $f_{ij}$ to $f_{-i,-j}$, a double-backward interpolation formula results.

These semi-forward and double backward interpolation formulae of $f_{ij}$ thus obtained may be proved to give the same results as those obtained from the corresponding interpolation formulae in terms of double differences.

Any semi-central interpolation formula, for example central $x$ and forward $y$, should make the full use of the tabular entries of both $f_{ij}$ and $f_{-i,j}$. One of the simplest ways of doing this is the following:

$$f(\pm u, v) = \tfrac{1}{2}[f(u, v) + f(-u, v)].$$

This semi-central interpolation formula can be proved to be correct by the same procedure as employed by Salzer[1,2] in proving the double forward interpolation formula, provided one can change $f(u, v)$ into $f(u, -v)$ whenever $v < 0$, and $f(u, v)$ into $f(-u, v)$ whenever $u < 0$. The other three semi-central interpolation formulae can be obtained similarly. Finally, the double-central interpolation formula may be written as follows:

$$f(\pm u, \pm v) = \tfrac{1}{4}[f(u, v) + f(u, -v) + f(-u, v) + f(-u, -v)].$$

Even though the interpolations made by using these semi-central and double-central interpolation formulae are very close to those calculated from the double interpolation formulae involving Stirling and Bessel central differences, the results are not exactly the same. Since the interpolation formulae of $f_{ij}$ use the tabular entries more symmetrically than those formulae of central differences do, it may be expected that the interpolation formulae of $f_{ij}$ give more convergent results.

**3. Double Differentiation Formulae.** In solving a differential equation numerically, either by an iterative method or by the step-ahead method, all the derivative terms of the differential equation should be expressed in terms of finite differences. Formulae for expressing the derivatives of different orders for one independent variable in terms of finite differences have been completely derived for forward, backward, and central differences.[3] There is thus no difficulty in replacing an ordinary differential equation by a difference equation.

However, the differential equations involved in solving most physical problems contain two or more independent variables. For solving partial differential equations with two independent variables numerically, formulae for expressing partial derivatives by double differences are required. In order to apply to all parts of a field of interest, the partial derivatives should be expressed by different combinations of double differences. These expressions have been derived and the results are given in Tables II, III, and IV.

In order to avoid the computations of double differences, it again may be worth while to work out some expressions of the partial derivatives in terms of tabular entries $f_{ij}$ directly instead of double differences. These expressions can be obtained from the corresponding double interpolation formulae of $f_{ij}$ previously derived. This is done by differentiating the interpolation formulae of $f_{ij}$ with respect to $u$ and $v$ the required number of times according to the kind and order of the partial derivative of interest and then letting $u = 0$ and $v = 0$. A general expression for the partial derivative of $f(u, v)$ of order $p + q$ is obtained as follows, where zeros are to be factored out in accordance with usual continuity convention:

$$\text{(2)}\quad \left[\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(u, v)\right]_{u=0,\, v=0} = \sum_{i+j=0}^{n} \sum_{a+b=p} \left\{ \frac{p!}{a!b!} \left[ \binom{0}{i} \sum_{i_\alpha=0}^{i-1}{}' \prod_{\alpha=1}^{a} \frac{1}{-i_\alpha} \right] \sum_{l+m=q} \frac{q!}{l!m!} \right.$$

$$\left. \times \left[ (-1)^{l+b} \binom{n}{n-i-j} \sum_{i_\alpha=0}^{n-i-j-1}{}' \prod_{\alpha=1}^{l+b} \frac{1}{n-i_\alpha} \right] \left[ \binom{0}{j} \sum_{i_\alpha=0}^{j-1}{}' \prod_{\alpha=1}^{m} \frac{1}{-i_\alpha} \right] \right\} f_{ij}$$

where $p$ and $q$ can be any integer. "$'$" on the summation sign indicates that no two $i_\alpha$ terms can be identical. For instance, when $i = 3$ and $a = 2$

$$\sum_{i_\alpha=0}^{2}{}' \prod_{\alpha=1}^{2} \frac{1}{-i_\alpha} = \frac{1}{0}\frac{1}{-1} + \frac{1}{0}\frac{1}{-2} + \frac{1}{-1}\frac{1}{0} + \frac{1}{-1}\frac{1}{-2} + \frac{1}{-2}\frac{1}{0} + \frac{1}{-2}\frac{1}{-1}.$$

The general expression of the derivative of $f(u, -v)$ is similar to equation (2) except that $(-1)^l$ is removed and a $(-1)^m$ term appears in front of $\binom{0}{j}$ and also $f_{ij}$ changes to $f_{i,-j}$. Similarly, the expression for the partial derivative of $f(-u, v)$ can be obtained from equation (2) by removing $(-1)^b$ and placing a $(-1)^a$ term in front of $\binom{0}{i}$, and at the same time changing $f_{ij}$ to $f_{-i,j}$. If both of these processes are carried out, an expression of the partial derivative of $f(-u, -v)$ is obtained. The partial derivative of $f(\pm u, v)$ then takes the form

$$\left[\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(\pm u, v)\right]_{00} = \frac{1}{2}\left[\left(\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(u, v)\right)_{00} + \left(\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(-u, v)\right)_{00}\right]$$

and

$$\left[\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(\pm u, \pm v)\right]_{00} = \frac{1}{4}\left[\left(\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(u, v)\right)_{00} + \left(\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(u, -v)\right)_{00} + \left(\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(-u, v)\right)_{00} + \left(\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} f(-u, -v)\right)_{00}\right].$$

TABLE II. *Derivatives in terms of forward and backward differences*

| | Forward $x$, Forward $y$ | Forward $x$, Backward $y$* | Backward $x$, Backward $y$ |
|---|---|---|---|
| $h\left(\frac{\partial f}{\partial x}\right)_{x=x_0,\,y=y_0}$ | $\Delta_x f_{00} - \frac{1}{2}\Delta_x^2 f_{00} + \frac{1}{3}\Delta_x^3 f_{00} - \frac{1}{4}\Delta_x^4 f_{00} + \cdots$ | $\Delta_x f_{00} - \frac{1}{2}\Delta_x^2 f_{00} + \frac{1}{3}\Delta_x^3 f_{00} - \frac{1}{4}\Delta_x^4 f_{00} + \cdots$ | $-\nabla_x f_{00} + \frac{1}{2}\nabla_x^2 f_{00} - \frac{1}{3}\nabla_x^3 f_{00} + \frac{1}{4}\nabla_x^4 f_{00} + \cdots$ |
| $k\left(\frac{\partial f}{\partial y}\right)_{00}$ | $\Delta_y f_{00} - \frac{1}{2}\Delta_y^2 f_{00} + \frac{1}{3}\Delta_y^3 f_{00} - \frac{1}{4}\Delta_y^4 f_{00} + \cdots$ | $-\nabla_y f_{00} + \frac{1}{2}\nabla_y^2 f_{00} - \frac{1}{3}\nabla_y^3 f_{00} + \frac{1}{4}\nabla_y^4 f_{00} + \cdots$ | $-\nabla_y f_{00} + \frac{1}{2}\nabla_y^2 f_{00} - \frac{1}{3}\nabla_y^3 f_{00} + \frac{1}{4}\nabla_y^4 f_{00} + \cdots$ |
| $h^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{00}$ | $\Delta_x^2 f_{00} - \Delta_x^3 f_{00} + \frac{11}{12}\Delta_x^4 f_{00} - \cdots$ | $\Delta_x^2 f_{00} - \Delta_x^3 f_{00} + \frac{11}{12}\Delta_x^4 f_{00} + \cdots$ | $\nabla_x^2 f_{00} - \nabla_x^3 f_{00} + \frac{11}{12}\nabla_x^4 f_{00} + \cdots$ |
| $hk\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{00}$ | $\Delta_x\Delta_y f_{00} - \frac{1}{2}\Delta_x^2\Delta_y f_{00} - \frac{1}{2}\Delta_x\Delta_y^2 f_{00} + \frac{1}{3}\Delta_x^3\Delta_y f_{00} + \frac{1}{3}\Delta_x\Delta_y^3 f_{00} + \frac{1}{4}\Delta_x^2\Delta_y^2 + \cdots$ | $-\Delta_x\nabla_y f_{00} + \frac{1}{2}\Delta_x^2\nabla_y f_{00} + \frac{1}{2}\Delta_x\nabla_y^2 f_{00} - \frac{1}{3}\Delta_x^3\nabla_y f_{00} - \frac{1}{4}\Delta_x^2\nabla_y^2 f_{00} - \frac{1}{3}\Delta_x\nabla_y^3 f_{00} + \cdots$ | $\nabla_x\nabla_y f_{00} - \frac{1}{2}\nabla_x^2\nabla_y f_{00} - \frac{1}{2}\nabla_x\nabla_y^2 f_{00} + \frac{1}{3}\nabla_x^3\nabla_y f_{00} + \frac{1}{3}\nabla_x\nabla_y^3 f_{00} + \frac{1}{4}\nabla_x^2\nabla_y^2 f_{00}$ |
| $k^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\Delta_y^2 f_{00} - \Delta_y^3 f_{00} + \frac{11}{12}\Delta_y^4 f_{00} - \cdots$ | $\nabla_y^2 f_{00} - \nabla_y^3 f_{00} + \frac{11}{12}\nabla_y^4 f_{00} + \cdots$ | $\nabla_y^2 f_{00} - \nabla_y^3 f_{00} + \frac{11}{12}\nabla_y^4 f_{00} + \cdots$ |
| $h^3\left(\frac{\partial^3 f}{\partial x^3}\right)_{00}$ | $\Delta_x^3 f_{00} - \frac{3}{2}\Delta_x^4 f_{00} + \cdots$ | $\Delta_x^3 f_{00} - \frac{3}{2}\Delta_x^4 f_{00} + \cdots$ | $-\nabla_x^3 f_{00} + \frac{3}{2}\nabla_x^4 f_{00} + \cdots$ |
| $h^2k\left(\frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial y}\right)_{00}$ | $\Delta_x^2\Delta_y f_{00} - \Delta_x^3\Delta_y f_{00} - \frac{1}{2}\Delta_x^2\Delta_y^2 f_{00} + \cdots$ | $-\Delta_x^2\nabla_y f_{00} + \Delta_x^3\nabla_y f_{00} + \frac{1}{2}\Delta_x^2\nabla_y^2 f_{00} + \cdots$ | $-\nabla_x^2\nabla_y f_{00} + \nabla_x^3\nabla_y f_{00} + \frac{1}{2}\nabla_x^2\nabla_y^2 f_{00} + \cdots$ |
| $hk^2\left(\frac{\partial}{\partial x}\frac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\Delta_x\Delta_y^2 f_{00} - \Delta_x\Delta_y^3 f_{00} - \frac{1}{2}\Delta_x^2\Delta_y^2 f_{00} + \cdots$ | $\Delta_x\nabla_y^2 f_{00} - \Delta_x\nabla_y^3 f_{00} + \frac{1}{2}\Delta_x^2\nabla_y^2 f_{00}$ | $-\nabla_x\nabla_y^2 f_{00} + \nabla_x\nabla_y^3 f_{00} + \frac{1}{2}\nabla_x^2\nabla_y^2 f_{00} + \cdots$ |
| $k^3\left(\frac{\partial^3 f}{\partial y^3}\right)_{00}$ | $\Delta_y^3 f_{00} - \frac{3}{2}\Delta_y^4 f_{00} + \cdots$ | $-\nabla_y^3 f_{00} + \frac{3}{2}\nabla_y^4 f_{00} + \cdots$ | $-\nabla_y^3 f_{00} + \frac{3}{2}\nabla_y^4 f_{00} + \cdots$ |
| $h^4\left(\frac{\partial^4 f}{\partial x^4}\right)_{00}$ | $\Delta_x^4 f_{00} + \cdots$ | $\Delta_x^4 f_{00} + \cdots$ | $\nabla_x^4 f_{00} + \cdots$ |
| $h^3k\left(\frac{\partial^3}{\partial x^3}\frac{\partial f}{\partial y}\right)_{00}$ | $\Delta_x^3\Delta_y f_{00} + \cdots$ | $-\Delta_x^3\nabla_y f_{00} + \cdots$ | $\nabla_x^3\nabla_y f_{00} + \cdots$ |
| $h^2k^2\left(\frac{\partial^2}{\partial x^2}\frac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\Delta_x^2\Delta_y^2 f_{00} + \cdots$ | $\Delta_x^2\nabla_y^2 f_{00} + \cdots$ | $\nabla_x^2\nabla_y^2 f_{00} + \cdots$ |
| $hk^3\left(\frac{\partial}{\partial x}\frac{\partial^3 f}{\partial y^3}\right)_{00}$ | $\Delta_x\Delta_y^3 f_{00} + \cdots$ | $-\Delta_x\nabla_y^3 f_{00} + \cdots$ | $\nabla_x\nabla_y^3 f_{00} + \cdots$ |
| $k^4\left(\frac{\partial^4 f}{\partial y^4}\right)_{00}$ | $\Delta_y^4 f_{00} + \cdots$ | $\nabla_y^4 f_{00} + \cdots$ | $\nabla_y^4 f_{00} + \cdots$ |

* Remaining semi-backward interpolation formula can be obtained by interchanging $x$ and $y$ as well as $h$ and $k$.

TABLE III. *Derivatives in terms of Central x and Forward y Differences*

Semi-central Difference

$k^4\left(\frac{\partial^4 f}{\partial y^4}\right)_{00}$ $\Delta_y^4 f_{00} + \cdots$ $\nabla_y^4 f_{00} + \cdots$ $\nabla_y^4 f_{00} + \cdots$

* Remaining semi-backward interpolation formula can be obtained by interchanging $x$ and $y$ as well as $h$ and $k$.

TABLE III. *Derivatives in terms of Central x and Forward y Differences*

| | Semi-central Difference | |
|---|---|---|
| | Stirling | Bessel |
| $h\left(\frac{\partial f}{\partial x}\right)_{00}$ | $\frac{\Delta_x f_{-1,0} + \Delta_x f_{00}}{2} - \frac{1}{6}\frac{\Delta_x^3 f_{-2,0} + \Delta_x^3 f_{-1,0}}{2} + \cdots$ | $\Delta_x f_{00} - \frac{1}{2}\frac{\Delta_x^2 f_{-1,0} + \Delta_x^2 f_{00}}{2} + \frac{1}{12}\Delta_x^3 f_{-1,0} + \frac{1}{12}\frac{\Delta_x^4 f_{-2,0} + \Delta_x^4 f_{-1,0}}{2} + \cdots$ |
| $k\left(\frac{\partial f}{\partial y}\right)_{00}$ | $\Delta_y f_{00} - \frac{1}{2}\Delta_y^2 f_{00} + \frac{1}{3}\Delta_y^3 f_{00} - \frac{1}{4}\Delta_y^4 f_{00} + \cdots$ | $\Delta_y f_{00} - \frac{1}{2}\Delta_y^2 f_{00} + \frac{1}{3}\Delta_y^3 f_{00} - \frac{1}{4}\Delta_y^4 f_{00} + \cdots$ |
| $h^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{00}$ | $\Delta_x^2 f_{-1,0} - \frac{1}{12}\Delta_x^4 f_{-2,0} + \cdots$ | $\frac{\Delta_x^2 f_{-1,0} + \Delta_x^2 f_{00}}{2} - \frac{1}{2}\Delta_x^3 f_{-1,0} - \frac{1}{12}\frac{\Delta_x^4 f_{-2,0} + \Delta_x^4 f_{-1,0}}{2} + \cdots$ |
| $hk\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{00}$ | $\frac{\Delta_x\Delta_y f_{-1,0} + \Delta_x\Delta_y f_{00}}{2} - \frac{1}{2}\frac{\Delta_x\Delta_y^2 f_{-1,0} + \Delta_x\Delta_y^2 f_{00}}{2}$ $- \frac{1}{6}\frac{\Delta_x^3\Delta_y f_{-2,0} + \Delta_x^3\Delta_y f_{-1,0}}{2} + \frac{1}{3}\frac{\Delta_x\Delta_y^3 f_{-1,0} + \Delta_x\Delta_y^3 f_{00}}{2} + \cdots$ | $\Delta_x\Delta_y f_{00} - \frac{1}{2}\frac{\Delta_x^2\Delta_y f_{-1,0} + \Delta_x^2\Delta_y f_{00}}{2} - \frac{1}{2}\Delta_x\Delta_y^2 f_{0,0} + \frac{1}{12}\Delta_x^3\Delta_y f_{-1,0}$ $+ \frac{1}{4}\frac{\Delta_x^2\Delta_y^2 f_{-1,0} + \Delta_x^2\Delta_y^2 f_{00}}{2} + \frac{1}{3}\Delta_x\Delta_y^3 f_{00} + \cdots$ |
| * $k^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\Delta_y^2 f_{00} - \Delta_y^3 f_{00} + \frac{11}{12}\Delta_y^4 f_{00} + \cdots$ | $\Delta_y^2 f_{00} - \Delta_y^3 f_{00} + \frac{11}{12}\Delta_y^4 f_{00} + \cdots$ |
| $h^3\left(\frac{\partial^3 f}{\partial x^3}\right)_{00}$ | $\frac{\Delta_x^3 f_{-2,0} + \Delta_x^3 f_{-1,0}}{2} + \cdots$ | $\Delta_x^3 f_{-1,0} - \frac{1}{2}\frac{\Delta_x^4 f_{-2,0} + \Delta_x^4 f_{-1,0}}{2} + \cdots$ |
| $h^2k\left(\frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial y}\right)_{00}$ | $\Delta_x^2\Delta_y f_{-1,0} - \frac{1}{2}\Delta_x^2\Delta_y^2 f_{-1,0} + \cdots$ | $\frac{\Delta_x^2\Delta_y f_{-1,0} + \Delta_x^2\Delta_y f_{00}}{2} - \frac{1}{2}\Delta_x^3\Delta_y f_{-1,0} - \frac{1}{2}\frac{\Delta_x^2\Delta_y^2 f_{-1,0} + \Delta_x^2\Delta_y^2 f_{00}}{2} + \cdots$ |
| $hk^2\left(\frac{\partial}{\partial x}\frac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\frac{\Delta_x\Delta_y^2 f_{-1,0} + \Delta_x\Delta_y^2 f_{00}}{2} - \frac{\Delta_x\Delta_y^3 f_{-1,0} + \Delta_x\Delta_y^3 f_{00}}{2} + \cdots$ | $\Delta_x\Delta_y^2 f_{00} - \frac{1}{2}\frac{\Delta_x^2\Delta_y^2 f_{-1,0} + \Delta_x^2\Delta_y^2 f_{00}}{2} - \Delta_x\Delta_y^3 f_{00} + \cdots$ |
| $k^3\left(\frac{\partial^3 f}{\partial y^3}\right)_{00}$ | $\Delta_y^3 f_{00} - \frac{3}{2}\Delta_y^4 f_{00} + \cdots$ | $\Delta_y^3 f_{00} - \frac{3}{2}\Delta_y^4 f_{00} + \cdots$ |
| $h^4\left(\frac{\partial^4 f}{\partial x^4}\right)_{00}$ | $\Delta_x^4 f_{-2,0} + \cdots$ | $\frac{\Delta_x^4 f_{-2,0} + \Delta_x^4 f_{-1,0}}{2} + \cdots$ |
| $h^3k\left(\frac{\partial^3}{\partial x^3}\frac{\partial f}{\partial y}\right)_{00}$ | $\frac{\Delta_x^3\Delta_y f_{-2,0} + \Delta_x^3\Delta_y f_{-1,0}}{2} + \cdots$ | $\Delta_x^3\Delta_y f_{-1,0} + \cdots$ |
| $h^2k^2\left(\frac{\partial^2}{\partial x^2}\frac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\Delta_x^2\Delta_y^2 f_{-1,0} + \cdots$ | $\frac{\Delta_x^2\Delta_y^2 f_{-1,0} + \Delta_x^2\Delta_y^2 f_{00}}{2} + \cdots$ |
| $hk^3\left(\frac{\partial}{\partial x}\frac{\partial^3 f}{\partial y^3}\right)_{00}$ | $\frac{\Delta_x\Delta_y^3 f_{-1,0} + \Delta_x\Delta_y^3 f_{00}}{2} + \cdots$ | $\Delta_x\Delta_y^3 f_{00} + \cdots$ |
| $k^4\left(\frac{\partial^4 f}{\partial y^4}\right)_{00}$ | $\Delta_y^4 f_{00} + \cdots$ | $\Delta_y^4 f_{00} + \cdots$ |

TABLE IV. *Derivatives in terms of Central Differences*

| | Central Differences | |
|---|---|---|
| | Stirling | Bessel |
| $h\left(\dfrac{\partial f}{\partial x}\right)_{00}$ | $\dfrac{\Delta_x f_{-1,0}+\Delta_x f_{00}}{2}-\dfrac{1}{6}\dfrac{\Delta_x^3 f_{-2,0}+\Delta_x^3 f_{-1,0}}{2}+\cdots$ | $\Delta_x f_{00}-\dfrac{1}{2}\dfrac{\Delta_x^2 f_{-1,0}+\Delta_x^2 f_{00}}{2}+\dfrac{1}{12}\Delta_x^3 f_{-1,0}+\dfrac{1}{12}\dfrac{\Delta_x^4 f_{-2,0}+\Delta_x^4 f_{-1,0}}{2}+\cdots$ |
| $k\left(\dfrac{\partial f}{\partial y}\right)_{00}$ | $\dfrac{\Delta_y f_{0,-1}+\Delta_y f_{00}}{2}-\dfrac{1}{6}\dfrac{\Delta_y^3 f_{0,-2}+\Delta_y^3 f_{0,-1}}{2}+\cdots$ | $\Delta_y f_{00}-\dfrac{1}{2}\dfrac{\Delta_y^2 f_{0,-1}+\Delta_y^2 f_{00}}{2}+\dfrac{1}{12}\Delta_y^3 f_{0,-1}+\dfrac{1}{12}\dfrac{\Delta_y^4 f_{0,-2}+\Delta_y^4 f_{0,-1}}{2}+\cdots$ |
| $h^2\left(\dfrac{\partial^2 f}{\partial x^2}\right)_{00}$ | $\Delta_x^2 f_{-1,0}-\frac{1}{12}\Delta_x^4 f_{-2,0}+\cdots$ | $\dfrac{\Delta_x^2 f_{-1,0}+\Delta_x^2 f_{00}}{2}-\dfrac{1}{2}\Delta_x^3 f_{-1,0}-\dfrac{1}{12}\dfrac{\Delta_x^4 f_{-2,0}+\Delta_x^4 f_{-1,0}}{2}+\cdots$ |
| $hk\left(\dfrac{\partial^2 f}{\partial x\partial y}\right)_{00}$ | $\dfrac{\Delta_x\Delta_y f_{-1,-1}+\Delta_x\Delta_y f_{-1,0}+\Delta_x\Delta_y f_{0,-1}+\Delta_x\Delta_y f_{00}}{4}$ $-\dfrac{1}{6}\dfrac{\Delta_x^3\Delta_y f_{-2,-2}+\Delta_x^3\Delta_y f_{-2,0}+\Delta_x^3\Delta_y f_{-1,-1}+\Delta_x^3\Delta_y f_{-1,0}}{4}$ $+\dfrac{\Delta_x\Delta_y^3 f_{-1,-2}+\Delta_x\Delta_y^3 f_{-1,-1}+\Delta_x\Delta_y^3 f_{0,-2}+\Delta_x\Delta_y^3 f_{0,-1}}{4}+\cdots$ | $\Delta_x\Delta_y f_{00}-\dfrac{\Delta_x^2\Delta_y f_{-1,0}+\Delta_x^2\Delta_y f_{00}+\Delta_x\Delta_y^2 f_{0,-1}+\Delta_x\Delta_y^2 f_{00}}{4}$ $+\dfrac{1}{4}\dfrac{\Delta_x^2\Delta_y^2 f_{-1,-1}+\Delta_x^2\Delta_y^2 f_{-1,0}+\Delta_x^2\Delta_y^2 f_{0,-1}+\Delta_x^2\Delta_y^2 f_{00}}{4}$ $+\dfrac{1}{6}\dfrac{\Delta_x\Delta_y^3 f_{0,-1}+\Delta_x^3\Delta_y f_{-1,0}}{2}+\cdots$ |
| $k^2\left(\dfrac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\Delta_y^2 f_{0,-1}-\frac{1}{12}\Delta_y^4 f_{0,-2}+\cdots$ | $\dfrac{\Delta_y^2 f_{0,-1}+\Delta_y^2 f_{00}}{2}-\dfrac{1}{2}\Delta_y^3 f_{0,-1}-\dfrac{1}{12}\dfrac{\Delta_y^4 f_{0,-2}+\Delta_y^4 f_{0,-1}}{2}+\cdots$ |
| $h^3\left(\dfrac{\partial^3 f}{\partial x^3}\right)_{00}$ | $\dfrac{\Delta_x^3 f_{-2,0}+\Delta_x^3 f_{-1,0}}{2}+\cdots$ | $\Delta_x^3 f_{-1,0}-\dfrac{1}{2}\dfrac{\Delta_x^4 f_{-2,0}+\Delta_x^4 f_{-1,0}}{2}+\cdots$ |
| $h^2k\left(\dfrac{\partial^2}{\partial x^2}\dfrac{\partial f}{\partial y}\right)_{00}$ | $\dfrac{\Delta_x^2\Delta_y f_{-1,-1}+\Delta_x^2\Delta_y f_{-1,0}}{2}+\cdots$ | $\dfrac{\Delta_x^2\Delta_y f_{-1,0}+\Delta_x^2\Delta_y f_{00}}{2}-\frac{1}{2}\Delta_x^3\Delta_y f_{-1,0}$ $-\dfrac{1}{2}\dfrac{\Delta_x^2\Delta_y^2 f_{-1,-1}+\Delta_x^2\Delta_y^2 f_{0,-1}+\Delta_x^2\Delta_y^2 f_{-1,0}+\Delta_x^2\Delta_y^2 f_{00}}{4}+\cdots$ |

TABLE IV—*Continued*

| | Central Differences | |
|---|---|---|
| | Stirling | Bessel |
| $hk^2\left(\frac{\partial}{\partial x}\frac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\frac{\Delta_x\Delta_y^2 f_{-1,-1} + \Delta_x\Delta_y^2 f_{0,-1}}{2} + \cdots$ | $\frac{\Delta_x\Delta_y^2 f_{0,-1} + \Delta_x\Delta_y^2 f_{00}}{2} - \frac{1}{2}\Delta_x\Delta_y^3 f_{0,-1}$ $- \frac{1}{2}\,\frac{\Delta_x^2\Delta_y^2 f_{-1,-1} + \Delta_x^2\Delta_y^2 f_{0,-1} + \Delta_x^2\Delta_y^2 f_{-1,0} + \Delta_x^2\Delta_y^2 f_{00}}{4} + \cdots$ |
| $k^3\left(\frac{\partial^3 f}{\partial y^3}\right)_{00}$ | $\frac{\Delta_y^3 f_{0,-2} + \Delta_y^3 f_{0,-1}}{2} + \cdots$ | $\Delta_y^3 f_{0,-1} - \frac{1}{2}\,\frac{\Delta_y^4 f_{0,-2} + \Delta_y^4 f_{0,-1}}{2} + \cdots$ |
| $h^4\left(\frac{\partial^4 f}{\partial x^4}\right)_{00}$ | $\Delta_x^4 f_{-2,0} + \cdots$ | $\frac{\Delta_x^4 f_{-2,0} + \Delta_x^4 f_{-1,0}}{2} + \cdots$ |
| $h^3k\left(\frac{\partial^3}{\partial x^3}\frac{\partial f}{\partial y}\right)_{00}$ | $\frac{\Delta_x^3\Delta_y f_{-2,-1} + \Delta_x^3\Delta_y f_{-1,-1} + \Delta_x^3\Delta_y f_{-2,0} + \Delta_x^3\Delta_y f_{-1,0}}{4} + \cdots$ | $\Delta_x^3\Delta_y f_{-1,0} + \cdots$ |
| $h^2k^2\left(\frac{\partial^2}{\partial x^2}\frac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\Delta_x^2\Delta_y^2 f_{-1,-1} + \cdots$ | $\frac{\Delta_x^2\Delta_y^2 f_{-1,-1} + \Delta_x^2\Delta_y^2 f_{-1,0} + \Delta_x^2\Delta_y^2 f_{0,-1} + \Delta_x^2\Delta_y^2 f_{00}}{4} + \cdots$ |
| $hk^3\left(\frac{\partial}{\partial x}\frac{\partial^3 f}{\partial y^3}\right)_{00}$ | $\frac{\Delta_x\Delta_y^3 f_{-1,-2} + \Delta_x\Delta_y^3 f_{-1,-1} + \Delta_x\Delta_y^3 f_{0,-2} + \Delta_x\Delta_y^3 f_{0,-1}}{4} + \cdots$ | $\Delta_x\Delta_y^3 f_{0,-1} + \cdots$ |
| $k^4\left(\frac{\partial^4 f}{\partial y^4}\right)_{00}$ | $\Delta_y^4 f_{0,-2} + \cdots$ | $\frac{\Delta_y^4 f_{0,-2} + \Delta_y^4 f_{0,-1}}{2} + \cdots$ |

TABLE V. *Partial Derivatives in terms of $f_{ij}$*

| | Forward $x$, Forward $y$ | Forward $x$, Backward $y$ | Backward $x$, Backward $y$ |
|---|---|---|---|
| $h\left(\dfrac{\partial f}{\partial x}\right)_{x=x_0,\,y=y_0}$ | $-\frac{11}{6}f_{00}+3f_{10}-\frac{3}{2}f_{20}+\frac{1}{3}f_{30}+\cdots$ | $-\frac{11}{6}f_{00}+3f_{10}-\frac{3}{2}f_{20}+\frac{1}{3}f_{30}+\cdots$ | $\frac{11}{6}f_{00}-3f_{-1,0}+\frac{3}{2}f_{-2,0}-\frac{1}{3}f_{-3,0}+\cdots$ |
| $k\left(\dfrac{\partial f}{\partial y}\right)_{00}$ | $-\frac{11}{6}f_{00}+3f_{01}-\frac{3}{2}f_{02}+\frac{1}{3}f_{03}+\cdots$ | $\frac{11}{6}f_{00}-3f_{0,-1}+\frac{3}{2}f_{0,-2}-\frac{1}{3}f_{0,-3}+\cdots$ | $\frac{11}{6}f_{00}-3f_{0,-1}+\frac{3}{2}f_{0,-2}-\frac{1}{3}f_{0,-3}+\cdots$ |
| $h^2\left(\dfrac{\partial^2 f}{\partial x^2}\right)_{00}$ | $2f_{00}-5f_{10}+4f_{20}-f_{30}+\cdots$ | $2f_{00}-5f_{10}+4f_{20}-f_{30}+\cdots$ | $2f_{00}-5f_{-1,0}+4f_{-2,0}-f_{-3,0}+\cdots$ |
| $hk\left(\dfrac{\partial^2 f}{\partial x\partial y}\right)_{00}$ | $2f_{00}-\frac{5}{2}f_{10}-\frac{5}{2}f_{01}+\frac{1}{2}f_{20}+3f_{11}$<br>$+\frac{1}{2}f_{02}-\frac{1}{2}f_{21}-\frac{1}{2}f_{12}+\cdots$ | $-2f_{00}+\frac{5}{2}f_{10}+\frac{5}{2}f_{0,-1}-\frac{1}{2}f_{20}-3f_{1,-1}$<br>$-\frac{1}{2}f_{0,-2}+\frac{1}{2}f_{2,-1}+\frac{1}{2}f_{1,-2}+\cdots$ | $2f_{00}-\frac{5}{2}f_{-1,0}-\frac{5}{2}f_{0,-1}+\frac{1}{2}f_{-2,0}+3f_{-1,-1}$<br>$+\frac{1}{2}f_{0,-2}-\frac{1}{2}f_{-2,-1}-\frac{1}{2}f_{-1,-2}+\cdots$ |
| $k^2\left(\dfrac{\partial^2 f}{\partial y^2}\right)_{00}$ | $2f_{00}-5f_{01}+4f_{02}-3f_{03}+\cdots$ | $2f_{00}-5f_{0,-1}+4f_{0,-2}-f_{0,-3}+\cdots$ | $2f_{00}-5f_{0,-1}+4f_{0,-2}-f_{0,-3}+\cdots$ |
| $h^3\left(\dfrac{\partial^3 f}{\partial x^3}\right)_{00}$ | $-f_{00}+3f_{10}-3f_{20}+f_{30}+\cdots$ | $-f_{00}+3f_{10}-3f_{20}+f_{30}+\cdots$ | $f_{00}-3f_{-1,0}+3f_{-2,0}-f_{-3,0}+\cdots$ |
| $h^2k\left(\dfrac{\partial^2}{\partial x^2}\dfrac{\partial f}{\partial y}\right)_{00}$ | $-f_{00}+2f_{10}+f_{01}-f_{20}-2f_{11}+f_{21}+\cdots$ | $f_{00}-2f_{10}-f_{0,-1}+f_{20}+2f_{1,-1}-f_{2,-1}+\cdots$ | $f_{00}-2f_{-1,0}-f_{0,-1}+f_{-2,0}+2f_{-1,-1}-f_{-2,-1}+\cdots$ |
| $hk^2\left(\dfrac{\partial}{\partial x}\dfrac{\partial^2 f}{\partial y^2}\right)_{00}$ | $-f_{00}+f_{10}+2f_{01}-f_{02}-2f_{11}+f_{12}+\cdots$ | $-f_{00}+f_{10}+2f_{0,-1}-f_{0,-2}-2f_{1,-1}+f_{1,-2}+\cdots$ | $f_{00}-f_{-1,0}-2f_{0,-1}+f_{0,-2}+2f_{-1,-1}-f_{-1,-2}+\cdots$ |
| $k^3\left(\dfrac{\partial^3 f}{\partial y^3}\right)_{00}$ | $-f_{00}+3f_{01}-3f_{02}+f_{03}+\cdots$ | $f_{00}-3f_{0,-1}+3f_{0,-2}-f_{0,-3}+\cdots$ | $f_{00}-3f_{0,-1}+3f_{0,-2}-f_{0,-3}+\cdots$ |

TABLE VI. *Partial derivatives in terms of $f_{ij}$*

| | Central $x$, Forward $y$ | Double Central |
|---|---|---|
| $h\left(\dfrac{\partial f}{\partial x}\right)_{x=x_0,\,y=y_0}$ | $\frac{3}{2}f_{10} - \frac{3}{2}f_{-1,0} - \frac{3}{4}f_{20} + \frac{3}{4}f_{-2,0} + \frac{1}{6}f_{30} - \frac{1}{6}f_{-3,0} + \cdots$ | $\frac{3}{2}f_{10} - \frac{3}{2}f_{-1,0} - \frac{3}{4}f_{20} + \frac{3}{4}f_{-2,0} + \frac{1}{6}f_{30} - \frac{1}{6}f_{-3,0} + \cdots$ |
| $k\left(\dfrac{\partial f}{\partial y}\right)_{00}$ | $-\frac{11}{6}f_{00} + 3f_{01} - \frac{3}{2}f_{02} + \frac{1}{3}f_{03} + \cdots$ | $\frac{3}{2}f_{01} - \frac{3}{2}f_{0,-1} - \frac{3}{4}f_{02} + \frac{3}{4}f_{0,-2} + \frac{1}{6}f_{03} - \frac{1}{6}f_{0,-3} + \cdots$ |
| $h^2\left(\dfrac{\partial^2 f}{\partial x^2}\right)_{00}$ | $2f_{00} - \frac{5}{2}f_{10} - \frac{5}{2}f_{-1,0} + 2f_{20} + 2f_{-2,0} - \frac{1}{2}f_{30} - \frac{1}{2}f_{-3,0} + \cdots$ | $2f_{00} - \frac{5}{2}f_{10} - \frac{5}{2}f_{-1,0} + 2f_{20} + 2f_{-2,0} - \frac{1}{2}f_{30} - \frac{1}{2}f_{-3,0} + \cdots$ |
| $hk\left(\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}\right)_{00}$ | $-\frac{5}{4}f_{10} + \frac{5}{4}f_{-1,0} + \frac{1}{4}f_{20} - \frac{1}{4}f_{-2,0} + \frac{3}{2}f_{11} - \frac{3}{2}f_{-1,1}$<br>$- \frac{1}{4}f_{21} + \frac{1}{4}f_{-2,1} - \frac{1}{4}f_{1,2} + \frac{1}{4}f_{-1,2} + \cdots$ | $\frac{3}{4}f_{11} - \frac{3}{4}f_{1,-1} - \frac{3}{4}f_{-1,1} + \frac{3}{4}f_{-1,-1} - \frac{1}{8}f_{21} + \frac{1}{8}f_{-2,1} + \frac{1}{8}f_{2,-1}$<br>$- \frac{1}{8}f_{-2,-1} - \frac{1}{8}f_{12} + \frac{1}{8}f_{-1,2} + \frac{1}{8}f_{1,-2} - \frac{1}{8}f_{-1,-2} + \cdots$ |
| $k^2\left(\dfrac{\partial^2 f}{\partial y^2}\right)_{00}$ | $2f_{00} - 5f_{01} + 4f_{02} - f_{03} + \cdots$ | $2f_{00} - \frac{5}{2}f_{01} - \frac{5}{2}f_{0,-1} + 2f_{02} + 2f_{0,-2} - \frac{1}{2}f_{03} - \frac{1}{2}f_{0,-3} + \cdots$ |
| $h^3\left(\dfrac{\partial^3 f}{\partial x^3}\right)_{00}$ | $\frac{3}{2}f_{10} - \frac{3}{2}f_{-1,0} - \frac{3}{2}f_{20} + \frac{3}{2}f_{-2,0} + \frac{1}{2}f_{30} - \frac{1}{2}f_{-3,0} + \cdots$ | $\frac{3}{2}f_{10} - \frac{3}{2}f_{-1,0} - \frac{3}{2}f_{2,0} + \frac{3}{2}f_{-2,0} + \frac{1}{2}f_{3,0} - \frac{1}{2}f_{-3,0} + \cdots$ |
| $h^2k\left(\dfrac{\partial^2}{\partial x^2}\dfrac{\partial f}{\partial y}\right)_{00}$ | $-f_{00} + f_{10} + f_{01} + f_{-1,0} - \frac{1}{2}f_{20} - \frac{1}{2}f_{-2,0} - f_{11}$<br>$- f_{-1,1} + \frac{1}{2}f_{21} + \frac{1}{2}f_{-2,1} + \cdots$ | $\frac{1}{2}f_{01} - \frac{1}{2}f_{0,-1} - \frac{1}{2}f_{11} + \frac{1}{2}f_{1,-1} - \frac{1}{2}f_{-1,1} + \frac{1}{2}f_{-1,-1}$<br>$+ \frac{1}{4}f_{21} - \frac{1}{4}f_{2,-1} + \frac{1}{4}f_{-2,1} - \frac{1}{4}f_{-2,-1} + \cdots$ |
| $hk^2\left(\dfrac{\partial}{\partial x}\dfrac{\partial^2 f}{\partial y^2}\right)_{00}$ | $\frac{1}{2}f_{10} - \frac{1}{2}f_{-1,0} - f_{11} + f_{-1,1} + \frac{1}{2}f_{12} - \frac{1}{2}f_{-1,2} + \cdots$ | $\frac{1}{2}f_{10} - \frac{1}{2}f_{-1,0} - \frac{1}{2}f_{11} - \frac{1}{2}f_{1,-1} + \frac{1}{2}f_{-1,1} + \frac{1}{2}f_{-1,-1}$<br>$+ \frac{1}{4}f_{12} + \frac{1}{4}f_{1,-2} - \frac{1}{4}f_{-1,2} - \frac{1}{4}f_{-1,-2} + \cdots$ |
| $k^3\left(\dfrac{\partial^3 f}{\partial y^3}\right)_{00}$ | $-f_{00} + 3f_{01} - 3f_{02} + f_{03} + \cdots$ | $\frac{3}{2}f_{01} - \frac{3}{2}f_{0,-1} - \frac{3}{2}f_{02} + \frac{3}{2}f_{0,-2} + \frac{1}{2}f_{03} - \frac{1}{2}f_{0,-3} + \cdots$ |

The expressions of the partial derivatives up to the third order have been written out explicitly as shown in Tables V and VI.

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[1] H. E. Salzer, *Jn. Math. and Physics*, v. 26, 1948, p. 294–305.
[2] H. E. Salzer, Amer. Math. Soc., *Bull.*, v. 51, 1945, p. 279–280.
[3] *Nautical Almanac*, 1937, p. 802.

## TECHNICAL NOTES AND SHORT PAPERS

### The First Published Table of Logarithms to the Base Ten

The first two lines of the title-page of this Table, *LOGARITHMORVM/ Chillias Prima.*, are followed by 26 further lines. The full title is given in James Henderson, *Biblioteca Tabularum Mathematicarum*, 1926, p. 30; a facsimile of the title-page is given in A. J. Thompson, *Logarithmetica Britannica*, part IX, 1924; or in his completed work, frontispiece to volume 2, 1954. There is nothing on the title-page to indicate the author, or the date and place of publication of the little 16-page pamphlet, which was, apparently, privately printed. On page 122 of John Ward, *The Lives of the Professors of Gresham College*, London, 1740, there is the following quotation from a letter, dated 6 December 1617, written by Sir Henry Bourchier to Dr. Usher: "Our kind friend Mr. Briggs hath lately published a supplement to the most excellent table of logarithms, which I presume he sent to you." Thus in connection with other facts the author was revealed to be Henry Briggs (1561–1631) and his table was printed in the latter part of 1617. John Napier (1550–1617) died in the previous April. Briggs had visited Napier in 1615, spent a month with him in 1616, and planned to show him this Table in the summer of 1617.

Copies of the Table are excessively rare; the only copies known to exist are two in the British Museum: (a) with press mark c.54e 10(1); (b) a copy in the Museum's Manuscript Room; and (c) a copy in the Savilian Library, Oxford. (a) and (c) are bound up with Edmund Gunter's *Canon Triangulorum*, 1623, so that the original may have been trimmed; its present size is $9.3 \times 15.5$ cm. A photostat copy of (a) is a recent acquisition of the Library of Brown University.

In this Table are given $\log N$, $N = [1(1)1000; 14D]$, with the first four numbers of first differences, rounded for $N = 500(1)1000$. Characteristics are separated from the decimal parts by lines. The accuracy of this Table is very extraordinary. Every entry of the Table was compared with A. J. Thompson's $\log N$, $N = [1(1)1000; 21D]$ and the only errors were 153 in the fourteenth decimal place: 150 unit errors, and 3 two-unit errors at $N = 154, 239, 863$.

In contrast to this, when we turn to Briggs' remarkable *Arithmetica Logarithmica*, London, 1624, giving $\log N$ for $N = [1(1)20\,000, 90\,000(1)10\,000; 14D]$ with difference throughout, we find, for $N = 1(1)1000$, no less than 19 errors; 18 two-units in the fourteenth decimal place and one serious error of 6 units in the seventh decimal place. Of the 150 unit errors in the 1617 publication the same

unit errors occur in 121 of the entries of the 1624 volume. In neither table did we make any checking of difference entries.

The idea of constructing a table in which the logarithm of unity was zero originated with Napier. Napier and Briggs never thought of logarithms as exponents of a base. An excellent exposition of their ideas is given in G. A. GIBSON, "Napier's logarithms and the change to Briggs's logarithms," p. 111–137 of C. G. KNOTT, *Napier Tercentenary Memorial Volume*, London, 1915; see also, H. S. CARSLAW, "The discovery of logarithms by Napier," *Mathematical Gazette*, v. 8, 1915, p. 76–84, 115–119. It was not till considerably later that our modern definition of a logarithm as an exponent was put forward by such mathematicians as DAVID GREGORY, 1684; WM. GARDINER, 1742; LEONARD EULER, 1748, 1770.

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## On the Numerical Integration of Functions Tabulated in Logarithmic Form*

In a number of physical problems which can be described by differential equations, it occurs that one or more of the variables show a range of several orders of magnitude. In such cases, it is convenient to tabulate the variables in logarithmic form. The purpose of this note is to show that the usual finite-difference formulae for numerical integration can be easily adapted to integrate a function when only its logarithm is given.

We shall limit our attention to the Newton-Gregory (backward-difference) formula, which is the most commonly used in the hand-integration of differential equations. Let $y'$ be the function to be integrated with respect to the independent variable $x$ between the limits $x_0 - h$ and $x_0$, where $h$ is the interval of tabulation. Writing $x = x_0 + hm$, we can approximate $\ln y'$ by the "Newton-backward" interpolating polynomial $f(m)$ as follows:

$$(1)\quad \ln y' = f(m) = f_0 + \binom{m}{1}\Delta'_{-\frac{1}{2}} + \binom{m+1}{2}\Delta''_{-1} + \cdots$$

$$= f_0 + \sum_{j=1}^{n}\binom{m+j-1}{j}\Delta^{(j)}_{-\frac{1}{2}j} + \text{truncation error.}$$

$$(j = 1, 2, 3, \cdots, n).$$

We now want to obtain an integration formula of the type

$$(2)\quad \ln\left(\frac{1}{h}\int_{x_0-h}^{x_0} y'dx\right) = \ln\int_{-1}^{0}\exp f(m)dm$$

$$= f_0 + \ln\int_{-1}^{0}\left\{1 + \sum_{j=1}^{n}\binom{m+j-1}{j}\Delta^{(j)}_{-\frac{1}{2}j}\right.$$

$$\left. + \frac{1}{2!}\left[\sum_{j=1}^{n}\binom{m+j-1}{j}\Delta^{(j)}_{-\frac{1}{2}j}\right]^2 + \cdots\right\} dm$$

$$= f_0 + N'\Delta'_{-\frac{1}{2}} + N''\Delta''_{-1} + \cdots.$$

Dropping the subscripts under the $\Delta^{(j)}$'s, which are always $-\frac{1}{2}j$, and making $n$ successively equal to 1, 2, etc., we have

$$\exp(N'\Delta') = \int_{-1}^{0} \exp(m\Delta')dm = \frac{1}{\Delta'}(1 - e^{-\Delta'}) = F',$$

(3)

$$\exp(N'\Delta' + N''\Delta'') = \int_{-1}^{0} \exp[m\Delta' + \tfrac{1}{2}(m^2 + m)\Delta'']dm = F'', \text{ etc.}$$

The individual coefficients $N^{(j)}$ can be computed from

$$N^{(j)}\Delta^{(j)} = \ln F^{(j)} - \ln F^{(j-1)}. \tag{4}$$

The evaluation of the coefficients $N^{(j)}$ in a power series of $\Delta^{(j)}$ is quite laborious for $j > 2$; it can be easily shown, however, that the first term of the series is always the corresponding Newton-Gregory coefficient:

$$N' = -\frac{1}{2} + \frac{1}{24}\Delta' - \frac{1}{2880}\Delta'^2 + \cdots$$

(5)

$$N'' = -\frac{1}{12} + \frac{1}{1440}\Delta'' + \frac{1}{720}\Delta'^2\Delta'' + \frac{1}{181440}\Delta''^2 + \cdots$$

$$N''' = -\frac{1}{24} + \cdots.$$

While the presence of the large term $+\frac{1}{24}\Delta'$ in the expression for $N'$ makes it necessary to tabulate this coefficient (or, rather, the product $N'\Delta'$) for practical work, it appears that the first term (i.e., the Newton-Gregory coefficient) is a sufficient approximation of $N^{(j)}$ for $j \geq 2$. The series for $N''$ shows that even in the extremely unlikely case of $\Delta' = \Delta'' = 1$, we have $N'' = -1/12.3$, and numerical integration of Equation (3) shows that when $\Delta'$, $\Delta''$, and $\Delta'''$ vary between 0 and 1, $N'''$ varies between $-1/24.0$ and $-1/24.5$. In practice, the interval of tabulation is always chosen such that $|\Delta'| \leq 1$ ($\Delta' = 1$ corresponds to an increment by a factor of $e$ in the variable from one tabular value to the next!), $|\Delta''| \leq 1/10$, $|\Delta'''| < |\Delta''|$, etc. Under such circumstances, the coefficients $N''$ and $N'''$ do not differ from the Newton-Gregory coefficients $-1/12$ and $-1/24$ by more than 1% and it stands to reason (although convergence proofs have not been undertaken) that the same will be true of the coefficients of the higher differences.

The method here described has been used with great success in the integration of the differential equations governing the motion and the loss of mass of meteors. Decimal, rather than natural, logarithms have been used throughout, and a small table of $N'\Delta'$ to four places, computed for the range from $-0.40$ to $+0.40$ in $\Delta'$ at 0.01-intervals has proved more than adequate. The table is not reproduced here, since it can be duplicated by any computer with very little effort.

The advantage of using the above formula should be quite obvious in the case of near-exponential functions, for which the step of integration can be made

much larger, without any increase in the truncation error, when the logarithm rather than the natural value of the function is tabulated.

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## Note on a Logarithm Algorithm

In a recent note,[1] D. SHANKS developed a well-known algorithm for the computation of logarithms,[2] in a way particularly suitable for use on automatic computing machines. In what follows I should like to point out that, if we use the value of $\mu = \ln a_0$ (the natural logarithm of $a_0$), the number of operations necessary for this computation can be cut down considerably in replacing about a third of the single steps indicated by Shanks by one division and one addition.

We assume as in the paper quoted that $a_0 > a_1 > 1$. To compute $\lambda = \log_{a_0} a_1$, we determine a sequence of numbers $a_2, a_3, a_4, \cdots, (a_i > 1)$ and a sequence of positive integers $n_1, n_2, \cdots$, by the relations

$$a_i^{n_i} \leq a_{i-1} < a_i^{n_i+1}, \quad a_{i+1} = a_{i-1}/a_i^{n_i} \quad (i = 1, 2, \cdots).$$

We then have

$$\lambda = \frac{1}{n_1+}\frac{1}{n_2+}\cdots;$$

if we stop at the calculation of $n_i$ we have an approximate value of $\lambda$ by taking the $i$th convergent:

$$\mu_i = \frac{P_i}{Q_i} = \frac{1}{n_1+}\frac{1}{n_2+}\cdots\frac{1}{n_i} = \lambda - \eta_i.$$

We will show that we have for $\eta_i$ the formula

$$\text{(I)} \qquad \eta_i = \eta_i^* + \rho_i, \quad \eta_i^* = (-1)^i\frac{a_{i+1} - 1}{\mu Q_i},$$

where the *error term* $\rho_i$ can be estimated by

$$\text{(II)} \qquad |\rho_i| \leq \mu Q_i\eta_i^2 \leq \mu|\eta_i|^{3/2} \leq \mu/Q_i^3$$

as soon as we have

$$\text{(III)} \qquad \mu/Q_i \leq 1.7933\cdots.$$

For $a_0 = 10$ we have $\mu = \mu_{10} \doteq 2.3026$ and (III) is certainly satisfied from $i = 2$ on. For $a_0 = 2$ we have

$$\mu = \mu_2 \doteq 0.6931.$$

Thus if we want, e.g., to compute $\log_2 a_1$ with an error less than $10^{-9}$ it is sufficient to stop with the computation of $n_i$ as soon as $Q_i$ exceeds $10^3$. In determining $n_i$ by successive division we will already have the value of $a_{i+1} = a_{i-1}/a_i^{n_i}$ and then apply (I).[3]

In order to prove (I)–(III), observe that we have for all positive integers

(1) $$a_{2\nu+1} = \frac{a_1^{Q_{2\nu}}}{a_0^{P_{2\nu}}}, \quad a_{2\nu} = \frac{a_0^{P_{2\nu-1}}}{a_1^{Q_{2\nu-1}}}.$$

Indeed, since $P_1 = 1, Q_1 = n_1, P_2 = n_2, Q_2 = n_1n_2 + 1$, our formulae are verified immediately for

$$a_2 = \frac{a_0}{a_1^{n_1}}, \quad a_3 = \frac{a_1}{a_2^{n_2}} = \frac{a_1^{n_1n_2+1}}{a_0^{n_2}}.$$

From there on we proceed by induction, using the well-known relations $P_i = n_iP_{i-1} + P_{i-2}, Q_i = n_iQ_{i-1} + Q_{i-2}$.

If we write now $\epsilon = (-1)^i$ we have from (1), for an even $i = 2\nu$:

$$a_{i+1} = e^{\mu\lambda Q_i - \mu P_i} = e^{\mu Q_i(\lambda - \mu_i)},$$

that is,

(2) $$a_{i+1} = e^{\epsilon\mu Q_i\eta_i};$$

this formula also holds in the case of an odd $i$, where we have $\epsilon = -1$.

From (2) we have now

(3) $$a_{i+1} - 1 = \epsilon\mu Q_i\eta_i + \theta(\mu Q_i\eta_i)^2,$$

where $\theta$ is the value of the series

$$\frac{e^\gamma - 1 - \gamma}{\gamma^2} = \sum_{\nu=0}^{\infty} \frac{\gamma^\nu}{(\nu + 2)!}, \quad \gamma = \epsilon\mu Q_i\eta_i.$$

Therefore we have certainly $|\theta| \leq 1$, if $|\gamma|$ does not exceed the positive root $\gamma_0$ of the equation

$$e^p = 1 + p + p^2;$$

we find

$$\gamma_0 \doteq 1.7933.$$

Assuming now that we have

(4) $$\mu Q_i|\eta_i| \leq \gamma_0,$$

we obtain (I), with $\rho_i = -\theta(\mu Q_i\eta_i^2)$, on dividing (3) by $\epsilon\mu Q_i$. The estimates (II) follow then immediately as we have $|\eta_i| \leq 1/Q_i^2$, $|\theta| \leq 1$. On the other hand, if (III) is satisfied, so is (4), since $Q_i|\eta_i| \leq 1/Q_i$.

We give for $a_0 = 10$, $a_1 = 2$ the exact values of the $\eta_i$ and the values $\eta_i^*$ computed by (I), in the last two columns of the following table for $i = 1, \cdots, 7$. The three first columns of this table contain the corresponding values of $Q_i$, $a_i$ and $\mu_i$. Although the values of the $a_i$ are given in the same example treated by

Shanks with 9 decimals for $i = 1, \cdots, 6$, we have recomputed them and computed further $a_7$ and $a_8$ on the $10 \times 10$ Friden Desk Calculator without double precision or any similar artifice.

In computing $P_i/Q_i + \eta_i^*$ it is better not to compute the second term separately from (I) but to use the formula

$$P_i/Q_i + \eta_i^* = \left[P_i + (-1)^i \frac{a_{i+1} - 1}{\mu}\right] \Big/ Q_i,$$

in replacing in this way two divisions by one only.

| $i$ | $Q_i$ | $a_i$ | $\mu_i$ | $\eta_i^*$ | $\eta_i$ |
|---|---|---|---|---|---|
| 1 | 3 | 2 | .33333 33333 333 | $-3.619 \times 10^{-2}$ | $-3.2 \times 10^{-2}$ |
| 2 | 10 | 1.25 | .30000 00000 000 | $1.042 \times 10^{-3}$ | $1.03 \times 10^{-3}$ |
| 3 | 93 | 1.024 | .30107 52688 1720 | $-4.549 \times 10^{-5}$ | $-4.527 \times 10^{-5}$ |
| 4 | 196 | 1.00974 1958 | .30102 04081 6327 | $9.6083 \times 10^{-6}$ | $9.5875 \times 10^{-6}$ |
| 5 | 485 | 1.00433 6278 | .30103 09278 3505 | $-9.32654 \times 10^{-7}$ | $-9.32171 \times 10^{-7}$ |
| 6 | 2136 | 1.00104 1546 | .30102 99625 468 | $3.312105 \times 10^{-8}$ | $3.311717 \times 10^{-8}$ |
| 7 | 13301 | 1.00016 2900 | .30102 99977 4453 | $-2.081453 \times 10^{-9}$ | $-2.0805493 \times 10^{-9}$ |
| 8 | | 1.00006 3748 | | | |

We see from this table that, while the error of $\mu_6$ is $3.3 \times 10^{-8}$, the improved method gives already $\mu_4 + \eta_4^*$ with the error $2.1 \times 10^{-8}$. On the other hand in using all computations which give $\mu_6$ we obtain at once $a_7$ and from there on obtain $\mu_6 + \eta_6^*$ by (I); this is in error by about $3.2 \times 10^{-12}$. If we go on to $i = 7$ the error of $\mu_7$ is $2.2 \times 10^{-9}$, while since we obtain at once $a_8$, we get from (I) the approximation $\mu_7 + \eta_7^*$ the error of which is about $9.0 \times 10^{-14}$. Here, however, the result is already influenced by the rounding off error in our value of $a_8$. If we use instead the value of $a_8$ obtained by double precision: 1.00006 37223 565, we obtain for $\eta_7^*$ the improved value $\bar{\eta}_7^* = -2.0806156 \times 10^{-9}$ and the error of $\mu_7 + \bar{\eta}_7^*$ is then $6.6 \times 10^{-14}$. We see that it is hardly worth-while to go over to double precision unless we need essentially more than 10 decimals.

As the essential steps in the original method consist in the successive determination of *integers*, it is clear that this method is not very sensitive with respect to the rounding off. It appears a little unexpected that the same is true to a very great extent for the improvement of this method proposed here. It turns out that the denominators $\mu Q_i$ in the formula (I) are sufficiently great to counterbalance to a certain extent the rounding off errors in the $a_i$.

It may be finally remarked that from a certain point of view the conclusions of D. Shanks concerning the linear character of convergence are perhaps slightly too optimistic. As a matter of fact the amount of work will not be measured by the number of "cycles," i.e., the number of partial quotients of the continued fraction used, but the total number of divisions; and this will be measured not by the number of the $n_i$ but by their sum $n_1 + n_2 + \cdots + n_i$. About this sum, however, KHINTCHINE has proved[4] that its order "in the average" is not that of $i$ but that of $i \lg i$. The "linearity of convergence" will probably fall off in the long run in the "general case."[5]

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[1] Daniel Shanks, "A Logarithm Algorithm," *MTAC*, v. 8, 1954, p. 60–64.
[2] See, e.g., J. Tropfke, *Geschichte der Elementar-Mathematik*. Bd. II: 3. Aufl., 1933, p. 241–242.
[3] In applying (I) we can of course replace the division by $\mu$ by the multiplication with $M = 1/\mu = \log_{a_0} e$, i.e., with the module of the logarithms to the base $a_0$. In this way we replace two divisions indicated in (I) by one division and one multiplication.
[4] A. Khintchine, *Metrische Kettenbruchprobleme. Comp. Math.*, v. 1, 1935, p. 360–382, especially p. 376, 377.
[5] Dr. D. Shanks to whom this note was submitted in manuscript made the very interesting remark that the use of $\mu = \ln a_0$ can be eliminated from my formula (I) in replacing $\mu$ by

$$P_i(a_i - 1) + P_{i-1}(a_{i+1} - 1) = \mu + O\left(\frac{1}{Q_{i-1}}\right).$$

However, in this case, the exact error estimate in (II) has to be slightly changed.

## Iterative Procedures for Taking Roots Based on Square Roots

**Introduction.** Several years ago, on the Model 1 C.P.C. at the Los Alamos Scientific Laboratory, there was incorporated by Dr. R. H. Stark a routine for taking square roots which proceeded at the same rate as the basic card feed operation. In connection with such an operation it is natural to ask if certain basic calculations might not be simplified by adding the square root as an operation. Today the existence of the Friden desk calculator, which also takes square roots directly, suggests that the question continues to have some cogency.

In connection with the Los Alamos C.P.C., we devised a procedure for taking cube roots and other roots which we present here for its conceivable usefulness and interest.

The Newton-Raphson iteration procedure for approximating roots of an equation $f(x) = 0$ establishes a function $g(x) = x - \dfrac{f(x)}{f'(x)}$ such that $x_{n+1} = g(x_n)$ gives an iterative procedure for finding roots starting with a guess $x_0$.

To take the cube root of a positive number $n$, then, we may consider equations $x^3 - n = 0$, $x^{3/2} - n^{1/2} = 0$, $x^{3/4} - n^{1/4} = 0$. The corresponding respective iteration functions are $g_1(x) = 1/3\left(\dfrac{n}{x^2} + 2x\right)$, $g_2(x) = 1/3(2n^{1/2}x^{-1/2} + x)$ and $g_3(x) = 1/3(4n^{1/4}x^{1/4} - x)$. Now either of the last two functions are not feasibly used unless one can readily take square roots or fourth roots. In general, they are more costly than the usual iteration $g_1(x)$. However, if one has no preference between, say, square roots and other arithmetic operations, then either $g_2(x)$ or $g_3(x)$ will give an iteration which tends to converge in fewer steps than the standard one. For example, we have found cases in which the second routine was adequate in four iterations, whereas the $g_1$ routine would require five starting with the same guess.

To handle negative roots one may assign the sign of $n$ to the first guess. We found it convenient to guess $\sqrt{n}$ in the C.P.C. which found $\sqrt{n}$ automatically. In general, one may take the seventh root of $n$, say, by using $g(x) = \dfrac{8(nx)^{1/8} - x}{7}$, and the ninth root by using $g(x) = 1/9\left(8\left(\dfrac{n}{x}\right)^{1/8} + x\right)$, and so on.

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## A Note on the Electronic Computer at Rothamsted

The "401" experimental electronic computer built by Elliott Brothers under contract from the National Research Development Corporation is now housed in the Statistical Department at Rothamsted, machine time being shared on a 50-50 basis between the N.R.D.C. and the Department. The Department is applying the computer to problems which arise in statistical research and its applications to agriculture, but time on the machine is available through the N.R.D.C. for general computations. The use of the computer specifically for the former work is a new venture. Prior to Rothamsted's acquiring the 401 little was known of the advantages and difficulties of applying electronic computers to statistics. Consequently, before being able to take full advantage of this powerful aid to computation, it was necessary to obtain experience not only of the appropriate programming techniques but also of the methods of adapting the rather special numerical techniques. By using a prototype computer the relevant experience is being gained and at the same time the characteristics that a permanent machine in this field ought to have are becoming apparent. This note gives a short description of the machine and an indication of the type of work being carried out.

The 401 is a serial computer with a word length of 32 binary digits and a word time of $100\mu s$. The main store is a magnetic disc of 23 tracks, each holding 128 words; 7 of the tracks are always immediately available to the computer, but only one of the remaining 16 is accessible at any given time, switching between these tracks being by means of high-speed relays. There are 5 single word immediate-access registers; one is the accumulator which can be coupled with the second register for double length working; the other 3 registers can each be used to modify orders as well as for temporary storage of numbers. Input is by 5-hole punched tape and output by either electrical typewriter or teleprinter-punch; punch-card input is shortly to be incorporated. The machine has a two-address code to allow for optimum programming. Operations possible include addition, subtraction, multiplication with and without round off, collation, non-equivalence, discrimination on zero or negative values, and left and right shifting of single or double length numbers. Addition and subtraction take one word time ($100\mu s$) to carry out, a shift of $n$ places takes $n$ word times and multiplication 32 word times (3.2 m.sec.).

A library of sub-routines is being built up and many of the standard functions and operations are now available. Routines for analysing randomised blocks and Latin square experiments have been used to obtain results for several large series of experiments. Work on routines for the analysis of factorial experiments is under way—these will allow a fast and accurate assessment of a wide variety of experiments. A programme has been developed which calculates the means, standard deviations, variances, covariances and correlation coefficients for samples of multivariate data. Specific problems tackled include a sampling investigation of the "combination of probabilities" test of significance: this necessitated a very large amount of computation and has been successfully concluded. General routines developed include a "programming" routine and a set of routines for floating binary operations. The former inserts the appropriate addresses in a skeleton programme and punches out the final routine for immediate use, thus

saving a great deal of programmer's time. The latter allows for the full 9 significant decimal figure accuracy and can accommodate numbers in the range $10^{\pm 8000}$.

The encouraging start made at Rothamsted seems to indicate that electronic computers are well suited for dealing with statistical problems, but undoubtedly new numerical and programming techniques are necessary to meet the special requirements of such work.

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## REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

This section contains reviews and descriptions of tables and of publications which in some explicit way have general interest in connection with tables or computation. Those noted as "Deposited in the UMT File" have been deposited in the Unpublished Mathematical Tables file maintained by the Chairman of the Editorial Committee; they are available there for reference. In many cases the author of these tables has a limited number of copies available for distribution. Authors of works containing mathematical tables of general interest or which otherwise fall into the classes noted or reviewed here are urged to submit a copy to the Chairman of the Editorial Committee.

31[A, F].—RSMTC *Table of Binomial Coefficients*. Royal Soc. Math. Tables, v. 3. Edited by J. C. P. Miller. Cambridge 1954. viii + 162 p. 21.5 × 28.0 cm. Price $6.50.

This table gives exact values of binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for positive integer values of $k$ and $n$. The main table (p. 2–103) is for $n \leq 200$. Because

$$\binom{n}{k} = \binom{n}{n-k}$$

it is unnecessary to print values for $k > n/2$. Actually $k$ is taken less than or equal to $(n + 1)/2$ plus whatever additional values of $k$ fill out the line. The central binomial coefficient in case $n$ is even, or the two central coefficients in case $n$ is odd, are printed in bold face type.

The rest of the table is for $n \leq 5000$ but for $k \leq L$ limited as follows

| $n$ | $L$ |
|---|---|
| 200–500 | 12 |
| 500–1000 | 11 |
| 1000–2000 | 5 |
| 2000–5000 | 3 |

Thus the table contains in particular the first 5000 triangular and tetrahedral numbers.

The arrangement of the table is such that the coefficients $\binom{n}{k}$ and $\binom{n}{k+1}$ are in general adjacent entries in the same line; that is, column headings are $k$. The opposite arrangement in which $\binom{n}{k}$ $(k = 0, 1, 2, \cdots)$ form a column of data would be perhaps more convenient for most purposes but would, no doubt, have rendered a difficult printing problem more difficult. On page 1 there is an index by means of which the reader can find any particular binomial coefficient.

The printing is by Cambridge University Press and is superlative, as always.

This table is intended primarily for number theorists whose interests in binomial coefficients are such that exact values are required. For the more practical man who wants no more than a half dozen significant digits the table will no doubt prove irritating in spite of the fact that the printer has supplied auxiliary figures to help him truncate the exact values and to express them in floating decimal notation. However the practical man may also have his curiosity aroused by inspection of the table. He may observe for example that for a fixed $n$ the number of odd binomial coefficients is always a power of 2. For such reasons the reviewer would recommend the table for general use. These are the most extensive tables of this function yet published. They are the result of many collaborating calculators.

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32[B].—ADMIRALTY RESEARCH LABORATORY, "Tables of $\eta = \sqrt{\{m^2 + \sqrt{-1}\}}$," A.R.L./T.7/Maths 2.7, April 1954, 4 p., Teddington, Middlesex, England.

The tabulation to six decimal places with second differences of $|\eta|$, $\arg \eta$, $R(\eta)$ and $I(\eta)$ is given for $m = 0(.02)3(.1)$ up to the point where $\eta = m + \frac{1}{8}m^{-3}$, $\arg \eta = \frac{1}{2}m^{-2}$, $R(\eta) = m + \frac{1}{8}m^{-3}$, $I(\eta) = \frac{1}{2}m^{-1}$, to the accuracy of the table.

I. A. STEGUN

NBSCL

33[D].—NBS *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*. Appl. Math. Ser. No. 36. U. S. Gov. Printing Office, Washington, 1953. x + 407 p. Price $3.00.

This is the third edition, published in 1953, of this set of tables originally published in 1939. See **RMT 89[D, E]**, *MTAC*, v. 1, 1943–45, p. 45, and Note 114, v. 4, 1950, p. 123. The present edition corrects two inconsistencies of format and six printing errors of the first edition; five of these printing errors have been corrected in the second edition.

The only other reported change is an extension of a supplementary table, which now expresses degrees, minutes, and seconds, in terms of radians to 10

decimal places and selected values of radians in terms of degrees, minutes, and seconds, to an accuracy of 0.000005 second.

C. B. T.

**34[D].**—L. W. POLLAK, *Eight-Place Supplement to Harmonic Analysis and Synthesis Schedules for Three to One Hundred Equidistant Values of Empiric Functions.* Second Edition, Dublin Institute for Advanced Studies, School of Cosmic Physics, Geophysical Memoirs No. 1, Dublin, 1954. Price £1 2s. 6d.

This is a second edition to the work issued in 1949, **RMT 847[K]** reviewed in *MTAC*, v. 5, 1951, p. 19–21. The work was written to be associated with "All term guide for harmonic analysis and synthesis," see **RMT 899[K]**, *MTAC*, v. 5, 1951, p. 149. However, directions are included for its use independently of the earlier "Guide."

In the present issue the accuracy of angles given in degrees, minutes, seconds, and fractions of a second, columns headed $iz$ and $\lambda z'$ has been carried to ten decimal places of a second. Some checks of accuracy of the earlier work are reported, and it is reported no mistakes were discovered.

C. B. T.

**35[E].**—AECD-3497, S. FRANKEL and E. NELSON, "Methods of treatment of displacement integral equations," Sept. 1953. Available from Office of Technical Services, Department of Commerce, Washington 25, D. C.

This contains on p. 78–82, a table of $k = k(f)$ defined by

$$-1 \le f \le 0, \qquad \frac{1}{1+f} = \frac{\operatorname{arctanh} k}{k}$$

$$0 \le f \le \infty, \qquad \frac{1}{1+f} = \frac{\arctan k}{k}$$

$f = -1(.01)2(.1)5(.5)10$. The table is to 6D, with first differences given.

A polynomial of degree 5 in $(1 + f)^{-1}$ which gives 4D in the range $8 < f < \infty$, is given.

J. T.

**36[K].**—LESLIE E. SIMON and FRANK E. GRUBBS, *Tables of the Cumulative Binomial Probabilities.* Ballistic Research Laboratories, Ordnance Corps. Pamphlet ORDP20-1, Aberdeen Proving Ground, 1952. 577 p. + viii, 12 × 9 inches, photo-offset from typescript. Available at U. S. Gov. Printing Office, Washington, D. C.

This table lists

$$P(c, n, p) = \sum_{r=c}^{n} \binom{n}{r} p^r (1-p)^{n-r}$$

for $c = 1(1) \cdots$ to $P < 10^{-7}$, $n = 1(1)150$, and $p = .01(.01)50.00$. This carries $n$ much farther than other available tables.

The introduction and foreword explain the use of the tables and give three illustrative examples. It is pointed out that the Incomplete Beta function $I_p(c, n - c + 1) = P(c, n, p)$ can be read from the table.

The type is small but is generously spaced: each value of $n$ has an integral number of pages for its own.

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37[L].—L. Fox, *A Short Table for Bessel Functions of Integer Order and Large Arguments*. Cambridge, 1954. Royal Society Shorter Mathematical Tables, No. 3. 28 p. Price $1.25. 27.8 × 21.7 cm.

The present tables are intended to supplement the tables of Bessel functions of integral order, $J_n$, $Y_n$, $I_n$ and $K_n$ previously prepared by the British Association Mathematical Tables Committee. In the earlier volumes the functions were tabulated for $n$ varying from 0 to 20 and $x$ in the range from 0 to 20 or 25. Here the range of $x$ extends from 20 to $\infty$. However, the tabular argument is $1/x$ so that use of the tables requires calculation of a reciprocal before entering. This device permits coverage of the range by means of fifty tabular values, namely $1/x = z = 0(.001).05$. In the case of $J_n(x)$, $Y_n(x)$ the usual auxiliary functions $P_n(x)$, $Q_n(x)$, for which

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}}\,[P_n(x)\cos(x - \tfrac{1}{2}n\pi - 1/4\pi) - Q_n(x)\sin(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)],$$

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}}\,[P_n(x)\sin(x - \tfrac{1}{2}n\pi - 1/4\pi) + Q_n(x)\cos(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)],$$

are tabulated. In the case of $I_n(x)$, $K_n(x)$, the auxiliary functions $F_n(x)$, $Y_n(x)$ such that

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}\,F_n(x), \quad K_n(x) \sim \sqrt{\frac{\pi}{2x}}\,e^{-x}G_n(x)$$

are tabulated. $P_n$ and $Q_n$ are given to nine decimals for $n = 0(1)9$ and to eight for $n \geq 10$. $F_n$ and $G_n$ are given to nine decimals for $n = 0(1)9$ while $\ln F_n$ and $\ln G_n$ are given to eight decimals for $n \geq 10$. Modified second and fourth central differences are given for interpolation.

$P_n$ and $Q_n$ were computed for $n = 0$ and $n = 1$ for $z = 0(.005).07$ and the recurrence relations

$$P_{n+1} - P_{n-1} = -2nzQ_n$$

$$Q_{n+1} - Q_{n-1} = 2nzP_n$$

were used to generate the values for those arguments for $n = 2(1)10$. The values were then subtabulated at intervals of .001 for $n = 0(1)10$. For $n > 9$ subtabulation was not always convenient and the values at intervals of .001 were generated by recurrence. Similar methods were used for the computation of $F_n$ and $G_n$. The tables were reproduced by photo-offset from a manuscript prepared on a

card-operated typewriter. From the numerous checks applied in the course of preparation one may feel confident that these tables are free from error.

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British Association Mathematical Tables, vol. 6. *Bessel Functions, Part I: Functions of orders zero and unity.* Cambridge University Press, 1937, reprinted 1950. [See **RMT 179[L]**, *MTAC*, v. 1, 1944, p. 361.]

British Association Mathematical Tables, vol. 10. *Bessel Functions, Part II: Functions of positive integer order.* Cambridge University Press, 1952. [See **RMT 1087[L]**, *MTAC*, v. 7, 1953, p. 97.]

**38[L].**—NBSCL, *Table of the Gamma Function for Complex Arguments.* NBS Applied Mathematics Series No. 34. U. S. Gov. Printing Office, Washington, D. C., 1954, xvi + 105 p., 22 × 20 cm. Price $2.00.

Values of the gamma function for complex arguments are required for work in atomic and nuclear physics, engineering, and elsewhere, and the publication of a reliable basic table of this function is to be welcomed. The present volume gives 12D values of the real and imaginary parts of $\log_e \Gamma(x + iy)$ for $x = 0(.1)10$, $y = 0(.1)10$. The values for negative $y$ may be obtained simply by changing the sign of the imaginary part, the values for negative $x$ from the relation $\Gamma(z)\Gamma(1 - z) = \pi \operatorname{cosec} \pi z$. To facilitate computation from this last formula, auxiliary tables of $\sin \pi x$, $\cos \pi x$, $\sinh \pi x$, $\cosh \pi x$ to 15D or 15S for $x = 0(.1)10$ are included. Thus, the tables given in this volume may be said to cover the region $-9 \leq x \leq 10$, $-10 \leq y \leq 10$. The effective range of the tables may be doubled in each direction by an application of the duplication formula; and outside the so increased range, the asymptotic expansion of $\log \Gamma(z)$ may be used.

Key values of $\log_e \Gamma(x + iy)$ for $x = 9(.1)10$, $y = 0(.1)10$ were computed to 15D from Stirling's expansion, the values for $0 \leq x \leq 8.9$ were obtained from the key values by application of the recurrence relation $\Gamma(z) = z^{-1}\Gamma(z + 1)$. A detailed description of the computation is given in the Introduction by H. E. SALZER which contains also notes on the purpose and scope of the table, on some properties of the gamma function, and on direct and inverse interpolation in the table. Also included is a bibliography listing 11 other tables of gamma functions for complex arguments, 16 auxiliary tables, and 16 references to books and papers containing relevant material on the gamma function.

The technical staff of the Mathematical Tables Project at the start of this undertaking consisted of A. N. LOWAN, Chief, MILTON ABRAMOWITZ, GERTRUDE BLANCH, ABRAHAM HILLMAN, WILLIAM HORENSTEIN, MEYER KARLIN, JACK LADERMAN, IDA RHODES, H. E. SALZER, and IRENE STEGUN. The key values for $9 \leq x \leq 10$ were computed by hand on desk calculators by RUTH CAPUANO, LEONA FREEMAN, ABRAHAM GROSSMAN, and RUTH ZUCKER. The extension to $0 \leq x \leq 8.9$ was done largely on punch-card machinery, by MILTON STEIN, who was also responsible for preparing and checking the final manuscript.

A. E.

**39[L].**—AKADEMIIA NAUK SSSR. [Institute of Exact Mechanics and Computational Techniques.] *Tablitsy Integralnogo sinusa i kosinusa* [*Tables of the sine and cosine integral*]. Moscow, 1954. 473 p. 25.7 × 19.7 cm. 43.75 rubles.

These tables give, to 7D,

$$\mathrm{Si}(x) = \int_0^x t^{-1} \sin t\,dt$$

$$\mathrm{Ci}(x) = \int_\infty^x t^{-1} \cos t\,dt$$

for $x = 0(.0001)2(.001)10(.005)100$.

There is a one page table of $C_i(x) - \ln x$ for $x = 0(.0001).0099$. These are rounded values taken from the 1940, 1942 NYMTP tables of these functions,[1] except that the range $10 < x < 100$, with the old interval of .01, has been subtabulated to .005. This was apparently done by hand, punched on cards, and checked with a tabulator. In most places no differences are given since linear interpolation suffices. When they are not negligible, second differences are given. A one page table of $\frac{1}{2}t(1 - t)$ and a nomogram for interpolating with $\Delta^2$ are supplied as inserts. The table is well printed on good quality paper. With good proof reading this could be a very accurate table. If one does not need the exponential integral, or more than 7D accuracy, this volume is a handy replacement for the three NYMTP volumes.

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[1] NYMTP. *Tables of Sine, Cosine and Exponential Integrals*. V. 1, 2. New York, 1940. *Table of Sine and Cosine Integrals*. New York, 1942.

40[L].—Akademiîa Nauk SSSR. [Institute of Exact Mechanics and Computational Techniques.] *Tabliîsy Integralov Frenelya* [*Tables of Fresnel Integrals*]. Moscow, 1953. 271 p. 16.2 × 25.8 cm. 23.50 rubles.

These fundamentally new tables of the Fresnel integrals

$$S(x) = \int_0^x \sin \tfrac{1}{2}\pi t^2 dt = \frac{1}{2}\int_0^x J_{\frac{1}{2}}(t)dt$$

$$C(x) = \int_0^x \cos \tfrac{1}{2}\pi t^2 dt = \frac{1}{2}\int_0^x J_{-\frac{1}{2}}(t)dt$$

give 7D values of these functions for

$$x = 0(.001)25$$

together with average second differences

$$\tfrac{1}{2}\{\Delta^2 f(x - \Delta x) + \Delta^2 f(x)\}.$$

Two short tables near $x = 0$ give 7S values (rather than the 7D values of the main table). These are tables of

$$S(x) \quad \text{for} \quad x = 0(.001).58$$

and

$$C(x) \quad \text{for} \quad x = 0(.001).101.$$

For interpolation there are two tables of $t(1-t)/2$ for $t = 0(.001).5$ and a nomogram for finding $\frac{1}{4}t(1-t)\Delta^2$.

The table was found by successive quadrature performed by Simpson's rule between the points $x = n$ and $x = n + 1$, $n = 0(1)24$. These 26 key values were computed from the power series and asymptotic formulas for $S$ and $C$.

The most extensive previous tables of the Fresnel integrals are those of van Wijngaarden & Scheen [*MTAC*, v. 4, p. 155] which appeared in 1949. These tables are to 5D for $x = 0(.01)20$.

The tables are well printed on good quality paper and represent a very considerable number of man hours of effort.

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**41[L].**—Admiralty Research Laboratory, "Table of $|I| = \left|\int_0^\varphi \sec\theta e^{i\mu\sec\theta}d\theta\right|$", A.R.L./T8/Maths 2.7, April 1954, 13 p., Teddington, Middlesex, England.

The function $|I|$ is tabulated to four decimal places for $\mu = 1(1)13$. The intervals in the table have been so chosen that linear interpolation in the $\varphi$ direction for the most part is adequate, or at most, the tabulated second differences must be used. In the region around $\varphi = 90°$, the auxiliary functions $h(\mu)$, $r$ and $\lambda$ are given, where $|I| = h(\mu)\{1 + 2r\cos\Omega + r^2\}^{\frac{1}{2}}$ and $\Omega$ (in degrees) $= \lambda + \dfrac{180}{\pi}\mu\sec\varphi$.

I. A. Stegun

NBSCL

**42[L].**—H. Gellman and Jean Tucker, "Tables of the functions $D_0(x)$ and $D_1(x)$." Atomic Energy of Canada Limited, Chalk River, Ontario, 1954. 51 p., 21.5 × 27.5 cm. $1.50.

The functions tabulated in this report are

$$D_0(x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta \exp(-x\cos\theta)$$

and $D_1(x) = -D_0'(x)$. They can be expressed in terms of modified Bessel functions and modified Struve functions as

$$D_0(x) = I_0(x) - \mathbf{L}_0(x), \quad D_1(x) = \frac{2}{\pi} - I_1(x) + \mathbf{L}_1(x).$$

$D_0(x)$ and $D_1(x)$ have been calculated on FERUT, the Ferranti Universal Digital Computor at the University of Toronto, and 7D values of them are given here for $x = 0(.001)4.999$.

A. E.

**43[L].**—M. ROTHMAN, "Tables of the integrals and differential coefficients of Gi$(+x)$ and Hi$(-x)$." *Quart. Jn. Mech. Appl. Math.*, v. 7, 1954. p. 379–384.

The functions

$$\mathrm{Gi}(x) = \pi^{-1}\int_0^\infty \sin\left(ux + \frac{u^3}{3}\right)du, \quad \mathrm{Hi}(-x) = \pi^{-1}\int_0^\infty \exp\left(-ux - \frac{u^3}{3}\right)du$$

were introduced by SCORER.[1] The present paper contains 7D tables of

$$\int_0^x \mathrm{Gi}(t)dt, \quad \frac{d\mathrm{Gi}(x)}{dx}, \quad \int_0^x \mathrm{Hi}(-t)dt, \quad \frac{d\mathrm{Hi}(-x)}{dx}$$

for $x = 0(.1)10$, with $\gamma_m{}^2$. The tables were obtained by numerical integration and differentiation of 10D tables of Gi$(x)$ and Hi$(-x)$, and the results were compared with those obtained from the asymptotic expansions for large $x$. The author has computed values up to $x = 20$, which are available if required.

A. E.

[1] R. S. SCORER, "Numerical evaluation of integrals of the form $I = \int_{x_1}^{x_2} f(x)e^{i\phi(x)}dx$ and the tabulation of the function $\mathrm{Gi}(z) = \frac{1}{\pi}\int_0^\infty \sin\left(uz + \frac{1}{3}u^3\right)du$," *Quart. Jn. Mech. Appl. Math.*, v. 3, 1950, p. 107–112; *MTAC*, v. 4, 1950, p. 215.

**44[L].**—M. ROTHMAN, *Table of the integrals of* Ai$(\pm x)$ *for* $x = 0.00(0.01)2.00$ *and* $x = 0.0(0.1)10.0$. i + 6 p. mimeographed. Deposited in the UMT FILE.

Ai$(x)$ is the Airy integral of the first kind, and the tables give

$$\int_0^x \mathrm{Ai}(t)dt$$

to 8D with $\delta_m{}^2$. The integrals were computed at intervals of 0.1 using the method of reduced derivatives. J. C. P. MILLER's unpublished table of reduced derivatives of Ai$(x)$ was used. A shortened version of this table has been published (see **RMT 1231**, *MTAC*, v. 8, 1954, p. 162).

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**45[L].**—M. ROTHMAN, *Table of* Bi$(+x)$ *for* $x = 0.00(0.01)2.00$ *and* Bi$(-x)$ *for* $x = 0.00(0.01)10.00$. i + 6 p. mimeographed. Deposited in the UMT FILE.

Bi$(x)$ is the Airy integral of the second kind, and the tables give 8D values of Bi$(\pm x)$ with $\delta_m{}^2$. The computation was carried out using the method of reduced derivatives and 12D were carried throughout the calculations. J. C. P. MILLER's manuscript tables of Bi$(x)$ and its reduced derivatives were utilized.

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**46[L].**—M. ROTHMAN, *Table of the integrals of* Bi($\pm x$) *for* $x = 0.00(0.01)2.00$ *and of* Bi($-x$) *for* $x = 0.0(0.1)10.0$. i + 6 p. mimeographed. Deposited in the UMT FILE.

8D tables, with $\delta_m^2$, of

$$\int_0^x \mathrm{Bi}(t)dt,$$

Bi($x$) being the Airy integral of the second kind. The integrals were computed at intervals of 0.1 using the method of reduced derivatives. J. C. P. MILLER's manuscript tables of reduced derivatives of Bi($x$) were used. The integrals were also computed from their ascending power series for $|x| \leq 2$.

A shortened version of this table has been published (**RMT 1231**, *MTAC*, v. 8, 1954, p. 162).

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**47[L].**—M. ROTHMAN, *Table of* Gi($x$) *and its derivative for* $x = 0.0(0.1)25(1)75$. i + 6 p. mimeographed. Deposited in the UMT FILE.

A shortened version of the table of Gi′($x$) has been published (see review 41 above). The mimeographed pages give 8D values of Gi($x$) and Gi′($x$) with $\delta_m^2$ (and for $x < 2.5$ also $\gamma^4$). The table of Gi′($x$) was obtained by numerical differentiation of a 10D table of Gi($x$).

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**48[L].**—M. ROTHMAN, *Table of* Hi($-x$) *and its derivative for* $x = 0.0(0.1)25(1)75$. i + 6 p. mimeographed. Deposited in the UMT FILE.

A shortened version of the table for Hi′($-x$) has been published (see review 41 above). The mimeographed pages give 8D values of Hi($-x$) and Hi′($-x$) with $\delta_m^2$ (and for $x < 2$ also $\gamma^4$). The table of Hi′($-x$) was obtained by numerical differentiation of a 10D table of Hi($-x$).

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**49[L].**—M. ROTHMAN, *Tables of the integrals of* Gi($x$) *and* Hi($-x$) *for* $x = 0.0(0.1)$-20.0. 4 p. mimeographed. Deposited in the UMT FILE.

A shortened version of these tables has been published (see review 41 above). The mimeographed pages give 8D values, with $\delta_m^2$ (and in a few places also $\gamma^4$), of

$$\int_0^x \mathrm{Gi}(t)dt, \quad \int_0^x \mathrm{Hi}(-t)dt.$$

The integrals were obtained by numerical integration of 10D tables of $\mathrm{Gi}(x)$ and $\mathrm{Hi}(x)$. They were checked from $x = 8$ onwards by comparison with results calculated from the asymptotic expansions.

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50[M].—TERENCE BUTLER and KARL POHLHAUSEN, *Tables of definite integrals involving Bessel functions of the first kind.* WADC Technical Report 54-420, Wright Air Development Center, 1954. 50 p. 21.5 × 27.5 cm.

$\gamma_r$ is the $r$th positive zero of $J_0(z)$. All values are tabulated to 5D.

Table I. $\gamma_r$ and $J_1(\gamma_r)$, $r = 1(1)10$. This table is taken from H. T. DAVIS and W. J. KIRKHAM, *Bull.* Amer. Math. Soc., v. 33, 1927, p. 760.

Table II. $J_\nu(\gamma_r)$, $r = 1(1)10$, $\nu = 1(1)\nu_r$, where $\nu_r$ is the last $\nu$ for which $J_\nu(\gamma_r)$ is not zero to 5D.

Tables III–VII.

$$\int_0^1 x^p J_0{}^m(\gamma_r x) J_1{}^n(\gamma_r x) dx,$$

where $p = 0(1)10$ and $r = 1(1)10$ in all tables; $m = 1$, $n = 0$ in Table III; $m = 0$, $n = 1$ in Table IV; $m = 2$, $n = 0$ in Table V; $m = 0$, $n = 2$ in Table VI; $m = n = 1$ in Table VII.

Tables VIII–X.

$$\int_0^1 x^p J_m(\gamma_r x) J_n(\gamma_s x) dx,$$

where $p = 0(1)5$, $r = 1(1)5$, $s = 1(1)5$ in all tables; $m = n = 0$ in Table VIII; $m = 0$, $n = 1$ in Table IX; and $m = n = 1$ in Table X.

A. E.

51[T, L].—ASCHER OPLER and NEVIN K. HIESTER, *Tables for Predicting the Performance of Fixed Bed Ion Exchange and Similar Mass Transfer Processes.* Stanford Research Institute, Stanford, California, 1954. Multilithed 111 p., 8½ × 11 inches. Free.

A variety of problems in the theory of non-equilibrium operation of ion-exchange and absorption columns in the theory of heat transfer and in the theory of probability may be described by the partial differential equation

$$-\left(\frac{\partial \lambda}{\partial s}\right)_t = \left(\frac{\partial \omega}{\partial t}\right)_s = \lambda(1 - \omega) - r\omega(1 - \lambda).$$

The solution of these equations may be written as

$$\lambda = \frac{J(rs, t)}{J(rs, t) + e^{(r-1)(t-s)}[1 - J(s, rt)]}$$

$$\omega = \frac{1 - J(t, rs)}{J(rs, t) + e^{(r-1)(t-s)}1 - J(s, rt)}$$

where

$$J(x, y) = 1 - \int_0^x e^{-y-\xi} I_0(2\sqrt{y\xi})d\xi.$$

The function $1 - J(x, y)$ has been previously calculated by BRINKLEY, EDWARDS, and SMITH (*MTAC*, v. 6, 1952, p. 40). However these calculations were found to be insufficient for the purposes of the authors when $J$ was near zero or one.

The evaluation of $\lambda$ and $\omega$ was carried out with an IBM 602-A using a modification on Onsager's asymptotic expansion.

Values are given to 4D and appear in three tables covering the following ranges.

| | | | |
|---|---|---|---|
| Table I: | $r = 0.2(0.2)1, 2$ | $s = 1, 2(2)8$ | $t/s = 0.2, 0.5, 1, 2, 5$ |
| Table II: | $r = 0.2(0.1)1, 1.2, 1.3, 1.4, 1.5, 2(1)5$ | | |
| | $s = 10(5)100(10)1{,}000$ | $t/s = 0.1(0.1)0.4(0.2)1(1)4, 6$ | |
| Table III: | $r = 0.2(.1).9$ | $s = 10(5)100(10)500$ | $t/s$ chosen so that $\lambda = 0.1$ and 0.9 |

The report contains the derivation of the differential equations in various problems and a description of the method of computation as well as the tables for $\lambda$ and $\omega$. In spite of a few typographical errors (e.g., $Z$ occurring in equation (9), p. 13 should be read as 2), this report is a useful collection of material for workers concerned with mass transfer problems.

A. H. T.

52[V].—AERONAUTICAL RESEARCH COUNCIL, *A Selection of Graphs for Use in Calculations of Compressible Airflow.* Prepared by the Compressible Flow Tables Panel (L. ROSENHEAD, Chairman, W. G. BICKLEY, C. W. JONES, L. F. NICHOLSON, H. H. PEARCEY, C. K. THORNHILL, R. C. TOMLINSON) of the ARC. Oxford, 1954. x + 115 p., $11\frac{1}{2}'' \times 15\frac{1}{4}''$, 84 s. net.

This book is a companion volume to "Compressible Flow: Tables" (Clarendon Press, 1952), **RMT 1093**[V], v. 7, 1953, p. 103. The preface states, "The object of both books is, briefly, to make available to engineers, physicists, and applied mathematicians a selection of tables and graphs likely to be of value in research and in calculations of the flow of air in which compressibility effects are important. It is hoped that the books will be useful both as an aid to design and for the development of new theory."

This handsomely printed and well thought-out book should succeed in fulfilling the hopes of the editors.

There are two types of graphs in this book, corresponding to single and double entry tables, and for both types single-page and multiple-page graphs are given. The single-page graphs are intended to give a quick appreciation of the variation of the quantities and they can also be used for rough working. The multiple page graphs corresponding to single entry tables enable one to take readings which approach those of the tables and in some cases are much more convenient to use than the tables.

In the mathematical functions plotted, $\gamma$, the ratio of specific heats of air has been taken to be 1.4 and $\mu$, the coefficient of viscosity for dry air at 23° C, to be

($183.00 \pm 0.25$) micropoise. This coefficient is equivalent to $(37^{4.1} \pm 0.5) \times 10^{-9}$ slug/ft. sec. at the standard sea level temperature of 15° C. The upper limit of the Mach number of the flow has been fixed arbitrarily at 5.

The graphs have been grouped in sections labelled A. Isentropic Flow, B. Normal Shocks, C. Oblique Shocks, D. Conical Flow, E. Reynolds Numbers. Each section contains its own introduction in which the basic equations used for the graphs are derived and briefly explained. The derivations do not always start with first principles. Nevertheless the equations plotted are clearly stated as are the figures in which the corresponding graphs are to be found.

This book will be a valuable and useful tool for the engineers, physicists, and applied mathematicians working in compressible airflow.

A. H. T.

53[V].—K. G. Tadman, *Tables of Flow Functions for Bodies of Revolution in Circular Tunnels and Jets*. Armament Research Establishment Memo 25/53 (modified). 11 mimeographed pages.

The functions defined by the following equations occur in the computation of irrotational incompressible flow patterns about bodies of revolution.

$$\phi(x, y, \nu, -1) + \frac{1}{2r} = \sum_{n=0}^{\infty} C_{\nu,n} \frac{r^n P_n(s)}{n!}$$

$$\phi(x, y, \nu, -2) + 1/2 \ln r(1 + s) = \sum_{n=0}^{\infty} C_{\nu,n-1} \frac{r^n P_n(s)}{n!}$$

$$1/2[r(1 + s)]^{-1} - \Omega(x, y, \nu, -2) = \sum_{n=0} C_{\nu,n-1} \frac{r^{n-1} P_n'(s)}{(n + 1)!}$$

$$1/4[\ln (r(1 + s)) - 1/2(1 - s)/(1 + s)] - \Omega(x, y, \nu, -3) = \sum_{n=0} C_{\nu,n-2} \frac{r^{n-1} P_n'(s)}{(n + 1)!}$$

where $r^2 = x^2 + y^2$, $s = x/r$, $P_n(s)$ is the Legendre polynomial of order $n$ and the $C_{\nu,n}$ are constants given in the memorandum to six significant figures.

This memorandum contains tables giving the values of the right hand sides of these equations to six significant figures for the case $\nu = 0$. These values have been used in an investigation into boundary corrections for axisymmetric cavities formed in cylindrical free jets. They are applicable only in the case of a boundary which is a free stream function.

The following tables are also included:

Table V: $-J_0(k_{0,m}y)$ Table VI: $-J_1(k_{0,m}y)/k_{0,m}y$

Table VII: $-\exp(-k_{0,m}x)/k_{0,m}J_1^2(k_{0,m})$

where $k_{\nu,m}$ is the $m$th positive zero of the Bessel function $J_\nu(z)$.

A. H. T.

54[Z].—*Proceedings of the First Conference on Training Personnel for the Computing Machine Field*. Edited by Arvid W. Jacobson, Wayne University Press, Detroit, 1944. 104 pages. Price $5.00.

On page 81 of this volume, L. W. Cohen makes two observations. "First, the effective use of the computing machine depends on the development of appropriate mathematical methods. Second, the development of mathematical methods depends on the development of mathematicians."

The operators of various computing machines seem to be becoming more aware of these dependences, and Wayne University has tried to make information available in a series of summer programs. This book reports the second of these programs. It is divided into four parts with headings, *Manpower Requirements in the Computer Field*, *Educational Programs*, *Influence of Automatic Computers on Technical and General Education*, and *Cooperative Efforts for Training and Research.* In the first section an appraisal of manpower requirements in business and industry, in government agencies, and by computer manufacturers, is made by three contributors. Each of these contributors encourages educational institutions to increase their training programs or states his opinion that there will be a tremendous demand for university graduates trained in the sciences which pertain to the computing machine field.

In the second section of the book several contributors spell out generalities of educational programs. In the third part, there is considerably more detail in what various universities are doing in their training programs. The Massachusetts Institute of Technology lists Machine-aided Analysis, Digital Computer Coding and Logic, Numerical Analysis, Methods of Applied Mathematics, a second course in Numerical Analysis, Switching Circuits, Digital Computer Applications and Practice, Analog Computation, Electronic Computation Laboratory, Pulsed-data Systems, and Switching Circuits, as courses directly related to automatic computation, for example.

It would seem well for universities engaged in a training program relating to computing machines or contemplating such a program to look through this book to extract from the first part some ideas about the potential demand, from the second part some ideas about the general aims and content of the program, from the third part some ideas concerning appropriate courses, and from the fourth part some ideas concerning the possibility of getting help in supporting such a program.

It is unfortunately true that the universities will have a hard time finding suitable text material for these courses and capable instructors, for the demands reported in the book are consuming the time and the efforts of people who might otherwise be available to write texts and to instruct in this field.

C. B. T.

## NOTES

### Summary of Educational Opportunities in Electronic Computation

A tabulated summary of educational opportunities in electronic computation is being prepared by Professor H. H. Goode, Professor of Electrical Engineering, University of Michigan. This Electronic Computation Education Summary will be published in the *Transactions* of the Professional Group on Electronic Computers of The Institute of Radio Engineers.

It is expected that the list will be revised from time to time as conditions change. Professor Goode is anxious to make the list as nearly complete as possible, and institutions with analog or digital computation courses or facilities, including punched card facilities, either planned or in operation for educational purposes, should communication directly with Professor Goode.

### Dr. E. W. Cannon Appointed Chief of the Applied Mathematics Division of The National Bureau of Standards

Dr. E. W. Cannon has been appointed Chief of the Applied Mathematics Division of the National Bureau of Standards, effective February 28, 1955.

He will direct the Bureau's basic mathematical research program which is directed principally toward better utilization of electronic computing machinery, and progress in numerical analysis, mathematical statistics, and mathematical physics. In addition, the Bureau's Applied Mathematics Division acts as a service laboratory to the Bureau and other Government agencies, particularly in the fields of numerical computation, statistics, and quality control.

### Policy Committee for Mathematics NBS Technical Advisory Committee

In its report of October 15, 1953, the Ad Hoc Committee, composed of representatives of professional scientific and engineering societies and under the chairmanship of M. J. Kelly, appointed by Secretary Weeks to review the activities of the National Bureau of Standards, recommended the formation of a set of Technical Advisory Committees to advise the Director of the Bureau of Standards and his staff on matters which the committees and the staff of the Bureau consider of importance. In accordance with this report known as the Kelly report, the Policy Committee of the Mathematical Societies of America, which was represented on the original Kelly Committee, nominated members for a Technical Advisory Committee for the Applied Mathematics Division of the Bureau of Standards.

At present this committee consists of David Blackwell of Howard University, Edward U. Condon, consulting physicist, Mark Kac of Cornell University, Phillip M. Morse of Massachusetts Institute of Technology, Mina Rees of Hunter College (Chairman), and A. H. Taub of the University of Illinois.

In meetings held on October 23, 1954, and on February 3, 1955, the Advisory Committee reviewed the program of the four sections of the Applied Mathematics Division, namely the Numerical Analysis Section, the Computation Laboratory, the Statistical Engineering Laboratory, and the Mathematical Physics Section. It also discussed that part of the work of the Electronic Computer Section having a bearing on the work of the Applied Mathematics Division.

The needs for new computing equipment and methods for financing the acquisition of such equipment were discussed at both meetings. A survey is being made by the Applied Mathematics Division of the computational needs of the other divisions of the Bureau of Standards, with two purposes in mind: 1) the determination of some of the requirements that the new computing equipment

must have, and 2) to acquaint the various divisions of the Bureau of Standards with the potentialities of modern computing devices and techniques as research tools.

The Applied Mathematics Division in particular and the Bureau of Standards in general provide unique services to the scientific and technical community. In turn they need the interest and support of this community in facing their financial, personnel, and scientific problems.

## The University of Wisconsin Conference "The Computing Laboratory in the University"

The University of Wisconsin is holding a conference entitled "The Computing Laboratory in the University," for a two and one half day period beginning Wednesday morning, August 17, 1955. There will be a few addresses on the computing field in general, several short talks by the ablest users of computing equipment, and several panel discussions concerned with the role of computing in higher educational institutions. The meeting is planned so as to be of interest to administrators and educators in higher educational institutions and to those in government agencies and industries who are responsibly concerned with the employment of trained personnel. Inquiries concerning the conference may be addressed to the Director of the Numerical Analysis Laboratory, 206 North Hall, The University of Wisconsin, Madison 6, Wisconsin.

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# CLASSIFICATION OF TABLES

A. Arithmetical Tables. Mathematical Constants

B. Powers

C. Logarithms

D. Circular Functions

E. Hyperbolic and Exponential Functions

F. Theory of Numbers

G. Higher Algebra

H. Numerical Solution of Equations

I. Finite Differences. Interpolation

J. Summation of Series

K. Statistics

L. Higher Mathematical Functions

M. Integrals

N. Interest and Investment

O. Actuarial Science

P. Engineering

Q. Astronomy

R. Geodesy

S. Physics, Geophysics, Crystallography

T. Chemistry

U. Navigation

V. Aerodynamics, Hydrodynamics, Ballistics

Z. Calculating Machines and Mechanical Computation

## CONTENTS

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